SEVENTH ANNUAL MEETING OF THE MISSOURI SECTION.

The seventh annual meeting of the Missouri Section was held at the University of Missouri, Columbia, Missouri, on Friday and Saturday, November 30 and December 1, 1923, in connection with the meeting of the Southwestern Section of the American Mathematical Society.

The attendance was thirty-two including the following twenty-one members of the Association: E. F. Allen, C. H. Ashton, H. Blumberg, W. C. Brenke, Theodosia T. Callaway, E. W. Chittenden, Mary E. Decherd, O. Dunkel, E. S. Haynes, E. R. Hedrick, L. Ingold, C. G. Jaeger, S. Lefschetz, J. V. McKelvey, U. G. Mitchell, P. R. Rider, E. Stephens, E. B. Stouffer, J. S. Turner, R. A. Wells, W. D. A. Westfall.

There was an enjoyable informal reception on Friday evening at the home of Professor E. R. Hedrick. There were two sessions of the Missouri Section on Saturday, both being presided over by the chairman, Professor Hedrick. The members of the two sections were the guests at a luncheon at the Daniel Boone Tavern on Saturday given by the Missouri Chapter of the Pi Mu Epsilon Fraternity. At the business meeting it was decided to hold the 1924 meeting in Kansas City at the time of the meeting of the State Teachers' Association. The following officers were elected for 1924: Chairman, R. R. FLEET, William Jewell College; Vice-chairman, R. A. WELLS, Park College; Secretary-Treasurer, P. R. RIDER, Washington University.

The following four papers were read:

(1) "Suggestions toward a comparative pedagogy of mathematics" (by invitation) by Professor HENRY BLUMBERG, University of Illinois.

(2) "An elementary discussion of the roots of the cubic" by Professor OTTO DUNKEL, Washington University.

(3) "A class of surfaces applicable to a sphere" by Mr. C. G. JAEGER, University of Missouri.

(4) "Service mathematics" by Professor THEODOSIA T. CALLAWAY, Stephens Junior College.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles.

1. The suggestions arise largely out of comparison of current mathematical pedagogy with pedagogic ideas and methods in music, art, literature, physics, biology, philosophy, etc. Professor Blumberg discussed these suggestions under three heads: (*a*) Initiation into mathematics. (*b*) Colleagueship of teacher and student. (*c*) Freedom from various accepted conceptions and prejudices. The paper is to appear in full in the future.

2. In this paper Professor Dunkel considers the cubic in which the second term has been removed and discovers the character of its roots by means of synthetic division, this process replacing the use of the derivative in some form

of treatment. In this manner the significance of the discriminant of the cubic is determined in an elementary manner.

3. In Mr. Jaeger's paper the equation of a two-parameter class of surfaces applicable to the sphere were obtained by solving Codazzi's equation after first making certain limitations on the second fundamental quantities, D, D', and D''.

4. Professor Callaway reported on service mathematics "a series of tests and exercises designated to locate and remove the mathematical difficulties of students in courses in elementary clothing and foods." A careful examination of standard texts in these subjects was made with a view to determining definitely what mathematical concepts and processes are used in such courses and the degree of difficulty to which each process is carried. The tests and exercises are based on the results of this study.

P. R. RIDER, *Secretary-Treasurer.*

THE ALGEBRA OF CORRELATION.

By DUNHAM JACKSON, University of Minnesota.

1. Introduction. The increased attention that is given to statistical methods by workers in the most diverse fields of science, as well as by mathematicians themselves, appears to be one of the most significant features of contemporary study. In a presentation of the mathematical aspects of the subject, there is room for considerable discretion as to the amount of mathematical knowledge to be demanded of the reader at the start. Even if the student finds a treatment suited to his preparation, he may be helped by a supplementary account which throws light on the matter from a slightly different angle, or clarifies the logical relations of the various parts by leading up to them in a slightly different order. It is on the basis of such considerations that the following exposition of some of the elementary theorems on correlation has been prepared. The reader who is acquainted with the existing literature of the subject will find nothing here that is new,[1] but he may find one item or another of his reading thrown into clearer relief. The mathematical prerequisites are covered by the ordinary freshman courses in trigonometry and college algebra. Some use will be made of the simplest parts of analytic geometry, but such facts as are needed in this connection, beyond the content of elementary courses in algebra, will be developed in the text.

2. Definition of the coefficient of correlation. Suppose that measurements have been made of two quantities pertaining to each of a set of n individuals of some sort. Let the observed values of the first quantity for the n individuals

[1] In particular, this article covers some of the same ground as the excellent paper of Professor Huntington in a recent volume of this MONTHLY, but the treatment is varied to such an extent that there is little actual duplication. The reader is referred to Professor Huntington's paper for a bibliography, which makes it unnecessary to include similar references here. See E. V. Huntington, "Mathematics and statistics, with an elementary account of the correlation coefficient and the correlation ratio," this MONTHLY (*1919*, 421–435).

respectively be $X_1, X_2, \cdots, X_n$, and let the corresponding values of the second quantity be $Y_1, Y_2, \cdots, Y_n$. The "individuals" may be persons, or groups of persons, or inanimate objects, or periods of time, etc.; the measured quantities may be the grades scored by the students of a class on a pair of tests, or the numbers of ballots cast in the counties of a state at a primary and at a regular election, or the lengths and breadths of a group of shells, or the amounts of rainfall and amounts of grain production in a given area in a series of years. To a reader experienced in the order of ideas involved, these vague indications will serve as an introduction to the definitions that are to be formulated. To one without such experience, they will probably convey little meaning. But that is of secondary consequence for the time being, inasmuch as a reader of either class will come into possession of all that is essential for the mathematical discussion by saying:

Let $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ *be any n pairs of real numbers.* They may be all distinct, or there may be repetitions among them; it will be assumed merely, for a reason which will be presently apparent, that the X's are not all equal to each other, and that the Y's are not all equal among themselves.

The *dramatis personæ* being thus assembled, let

$$\overline{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n), \qquad \overline{Y} = \frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n),$$

and for each value of k from $k = 1$ to $k = n$ let

$$x_k = X_k - \overline{X}, \qquad y_k = Y_k - \overline{Y},$$

so that the x's and y's are the deviations of the X's and Y's from their respective arithmetical means.

The *coefficient of correlation* between the X's and the Y's is defined by the formula

$$r = (\Sigma x_k y_k)/\sqrt{(\Sigma x_k^2)(\Sigma y_k^2)}, \tag{1}$$

the summation in each case extending[1] from $k = 1$ to $k = n$. In accordance with a convention which is practically universal, the radical sign by itself is understood to represent always the *positive* square root, a minus sign being written explicitly if the negative root is meant. The denominator of the expression for r is therefore always positive, and the algebraic sign of r is the same as the algebraic sign of the numerator.

It has been emphasized that the X's and the Y's, for the purpose in hand, may be any real numbers whatever, subject only to the condition that neither X_k nor Y_k is constant throughout. This condition is equivalent to saying that neither Σx_k^2 nor Σy_k^2 shall be zero, and is imposed for the sake of insuring that the denominator in the expression for r shall not vanish. The quantities x_k and y_k, on the other hand, are less thoroughly arbitrary, since it follows immediately from their definition that

$$\Sigma x_k = \Sigma y_k = 0. \tag{2}$$

[1] This understanding with regard to the sign of summation will be maintained throughout the paper.

Subject to this restriction, however, and to the condition that neither set shall consist exclusively of zeros, they may also be taken arbitrarily.[1]

The reader who is accustomed to work in a particular field of application may be prone to object, as the discussion proceeds, that an argument formulated in such general terms is too abstract to carry conviction. Such a critic is urged to bear in mind that the symbols employed represent *numbers*, and that the statement of an algebraic theorem is a statement that certain numerical relations will be verified by calculation in any specific instance. The assertion that *if you perform* certain arithmetical operations, *you will get* certain results, is a prediction with regard to a phenomenon at least as concrete as those with which many a psychological investigation is concerned; and if the calculation is performed by mechanical operations on a calculating machine, the verification is as objectively manifest to the senses as in any physical experiment. The pure mathematician may or may not be disposed to claim that his conclusions have a higher kind of authority than those of the experimental scientist,[2] but he can at any rate be confident that his results have every kind of substantial reality that belongs to any branch of knowledge.

3. Homogeneity of r. The value of the expression for r is unchanged[3] if x_k is replaced throughout by cx_k, c being positive and independent of k. For the numerator and the denominator are both multiplied by c, which cancels from the quotient. This property is of fundamental importance. It means that the units of measurement for the two sets of observed quantities can be chosen independently of each other. If the two sets of quantities are of the same kind, the units need not be the same in both cases;[4] and, what is more important, if the quantities are of different kinds, so that the units are not comparable at all, the coefficient nevertheless may have a definite meaning.[5]

4. Numerical limits of r. If it happens that the numbers x_k and y_k are so related that $y_k = cx_k$, for all values of k, where c is a positive number independent of k, it is seen that

$$\Sigma x_k y_k = c\Sigma x_k^2, \qquad \sqrt{\Sigma y_k^2} = c\sqrt{\Sigma x_k^2}, \qquad r = 1.$$

If $y_k = -cx_k$, the number c itself again being positive, then $\Sigma x_k y_k = -c\Sigma x_k^2$. On the other hand, as a result of the understanding that the radical sign always represents the positive square root,

$$\sqrt{\Sigma y_k^2} = +c\sqrt{\Sigma x_k^2}.$$

[1] No other restriction is necessary, because any numbers whose algebraic sum is zero may be regarded as constituting their own deviations from their own arithmetical mean.

[2] The present writer would be exceedingly reluctant to make any such claim.

[3] The same conclusion would hold if the corresponding expression were formed with X_k and Y_k instead of x_k and y_k; that is, it is independent of the restriction (2).

[4] For example, in correlating two sets of examination grades, it is not necessary to have the two examinations marked on the same scale.

[5] Of course the value of the coefficient will be affected by a change in the method of measurement of one of the quantities, such as the substitution of an area for a length in estimating the size of an object, or the assignment of different relative weights to the questions on an examination.

In this case, therefore, $r = -1$. The coefficient of correlation is plus or minus 1, according as y_k is positively or negatively proportional to x_k. In terms of the original quantities X_k and Y_k, the condition of proportionality means that

$$Y_k - \overline{Y} = \pm c(X_k - \overline{X}), \qquad Y_k = \pm cX_k + (\overline{Y} \mp c\overline{X}),$$

the last parenthesis being independent of k; that is, Y_k is a linear function of X_k.

It will be shown that if y_k is not proportional to x_k at all, the value of r is always *between* $+1$ and -1.

The proof will be made to depend on a well-known fact about the roots of a quadratic equation. Suppose an expression of the form $ax^2 + bx + c$ (for example, $x^2 + 2x + 2$) is written down with such coefficients that the value of the expression is always positive, no matter what (real) value is given to x. Since every real value of x makes the expression positive, no real value of x makes it zero. That is, the quadratic equation $ax^2 + bx + c = 0$ has no real root, and the discriminant $b^2 - 4ac$ must be negative.

Now suppose the numbers x_k and y_k are not proportional. If t is an arbitrary real number, the quantities $y_k - tx_k$ are not all zero for any one value of t, for their simultaneous vanishing would be precisely the condition of proportionality of y_k and x_k. Form the expression $\Sigma(y_k - tx_k)^2$. Being a sum of squares, this expression can never be negative. But it can also never be zero, for its terms can not all be zero simultaneously, and so some of them at least must be positive. If the expression is expanded in the form

$$\Sigma y_k^2 - 2t\Sigma x_k y_k + t^2\Sigma x_k^2,$$

it is seen to be *a quadratic expression in t, which is positive for all real values of t.* It comes under the rule of the preceding paragraph, if x, a, b, c are replaced by t, Σx_k^2, $(-2\Sigma x_k y_k)$, and Σy_k^2 respectively. So $4(\Sigma x_k y_k)^2 - 4(\Sigma x_k^2)(\Sigma y_k^2)$ must be a negative quantity; that is,

$$(\Sigma x_k y_k)^2 < (\Sigma x_k^2)(\Sigma y_k^2).$$

But the members of the last inequality are the squares of the numerator and the denominator respectively of the formula for r. It is recognized that $r^2 < 1$, and hence that r must be between [1] $+1$ and -1.

5. Correlation by rank. In an actual problem, the calculation of r may call for a considerable amount of numerical work. When the original data are not accurate enough to justify such an elaborate calculation, a formula is sometimes used which is considerably easier to evaluate. The individuals measured are arranged in order according to the observed values of X, the one with the largest X being placed first, the one with the next largest X, second, and so on. Then they are again arranged in order—generally, of course, in a different order—

[1] In this section, as in § 3, no use has been made of the condition (2). The proof and the conclusion apply equally well to the quantity which is obtained if x_k and y_k in the formula defining r are replaced by arbitrary numbers X_k and Y_k. That is, the quantity so obtained is equal to ± 1 if X_k and Y_k are proportional, and is between $+1$ and -1 in all other cases.

according to the observed values of Y. The difference between the rank-numbers assigned to an individual in the two arrangements is denoted by D_k, and the *coefficient of correlation by rank* is

$$\rho = 1 - 6(\Sigma D_k^2)/(n^3 - n). \tag{3}$$

It will now be shown that this is not really a new kind of correlation coefficient, but only a special case under the definition already given. Let X_1', $\cdots$, X_n', Y_1', $\cdots$, Y_n' be the rank-numbers in the two arrangements. That is, X_1' is equal to 1, 2, 3, $\cdots$, according as X_1 ranks first, second, third, $\cdots$, in order of algebraic magnitude, among all the numbers X_k, and so on.[1] It will appear that ρ is the coefficient of correlation between the numbers X_k' and the numbers Y_k', according to the definition [2] of § 2. To put it in another way, if ρ is allowed to denote this coefficient of correlation, it will be found that ρ is given by the formula (3), where $D_k = X_k' - Y_k'$.

Let x_k' and y_k' be the deviations of X_k' and Y_k' from their respective means. Then, by definition,

$$\rho = (\Sigma x_k' y_k')/\sqrt{(\Sigma x_k'^2)(\Sigma y_k'^2)}, \tag{4}$$

the relation (3) being temporarily in abeyance, as a theorem to be proved.

The numbers X_k' in the aggregate are merely the numbers 1, 2, 3, $\cdots$, n, arranged in some order. Their mean is consequently $\frac{1}{2}(n+1)$. This can be found by applying the formula for an arithmetical progression,

$$\frac{1}{n}(1 + 2 + 3 + \cdots + n) = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2},$$

or by pairing 1 with n, 2 with $n-1$, and so on, and taking the mean of each pair. The mean of the Y_k''s is the same, for the same reason. A first inference is that

$$D_k = X_k' - Y_k' = \left(x_k' + \frac{n+1}{2}\right) - \left(y_k' + \frac{n+1}{2}\right) = x_k' - y_k'.$$

The numbers x_k' are the numbers

$$n - \frac{n+1}{2}, \quad (n-1) - \frac{n+1}{2}, \quad \cdots, \quad 1 - \frac{n+1}{2}, \tag{5}$$

in some order, and the y_k''s are the same numbers, in the same or (generally) in a different order.

A further inference is that $\Sigma x_k'^2 = \Sigma y_k'^2$. Hence the two factors under the radical sign in (4) are equal, and, if the subscripts are omitted for simplicity,

$$\sqrt{(\Sigma x'^2)(\Sigma y'^2)} = \Sigma x'^2 = \Sigma y'^2 = \tfrac{1}{2}(\Sigma x'^2 + \Sigma y'^2),$$

[1] If two or more of the X_k's are tied, it is customary to divide the corresponding rank-numbers among the individuals concerned, using fractions if necessary; the theory is somewhat complicated thereby, and this case will be left aside.

[2] Hence, in particular, the extreme values of ρ are $+1$ and -1, as in any other application of the r-formula. As far as the limit $+1$ is concerned, this is also evident directly from (3).

the last expression being set down because of its use in the ensuing reductions. By a succession of steps, somewhat artificial in appearance, perhaps, but exceedingly simple, it is seen that

$$\rho = \frac{2\Sigma x'y'}{\Sigma x'^2 + \Sigma y'^2} = \frac{\Sigma x'^2 + \Sigma y'^2 - \Sigma x'^2 + 2\Sigma x'y' - \Sigma y'^2}{\Sigma x'^2 + \Sigma y'^2}$$

$$= 1 - \frac{\Sigma(x'^2 - 2x'y' + y'^2)}{\Sigma x'^2 + \Sigma y'^2} = 1 - \frac{\Sigma D^2}{\Sigma x'^2 + \Sigma y'^2} = 1 - \frac{\Sigma D^2}{2\Sigma x'^2}.$$

The resemblance to (3) is now apparent; it remains to evaluate the denominator $2\Sigma x'^2$.

It would not be a problem of excessive difficulty to find the sum of the squares of the numbers (5) directly. The work can be much simplified, however, by an observation which is important in other connections as well. Let X_k, $\overline{X}$, and x_k have the general significance assigned to them in § 2. Then [1]

$$\Sigma X_k^2 = \Sigma(x_k + \overline{X})^2 = \Sigma x_k^2 + 2\overline{X}\Sigma x_k + n\overline{X}^2,$$

or, *because of the fact that* $\Sigma x_k = 0$,

$$\Sigma x_k^2 = \Sigma X_k^2 - n\overline{X}^2.$$

In the present instance,

$$\Sigma x_k'^2 = \Sigma X_k'^2 - n\left(\frac{n+1}{2}\right)^2$$

$$= 1^2 + 2^2 + \cdots + n^2 - n\left(\frac{n+1}{2}\right)^2.$$

It is well known that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6};$$

the proof is a simple exercise in mathematical induction, regularly presented in courses in college algebra, and need not be detailed here. Hence

$$\Sigma x'^2 = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{4} = \frac{n^3 - n}{12} = \frac{n(n^2-1)}{12},$$

and

$$\rho = 1 - \frac{\Sigma D^2}{2\Sigma x'^2} = 1 - \frac{6\Sigma D^2}{n(n^2-1)}.$$

6. Spearman's Footrule. A still simpler formula involving rank-numbers is known as Spearman's Footrule.[2] In the notation of the preceding section, it is

$$R = 1 - \frac{3\Sigma|D_k|}{n^2 - 1},$$

[1] More generally, if h is any number whatever,

$$\Sigma(x_k + h)^2 = \Sigma x_k^2 + nh^2.$$

[2] C. Spearman, 'Footrule' for measuring correlation, *British Journal of Psychology*, vol. 2 (1906–08), pp. 89–108. This paper calls attention also to the formula of the preceding section.

the symbol $|D_k|$ denoting of course the absolute value[1] of D_k. This R will not be discussed here, except to the extent of showing that it is *not* a special case of the coefficient r defined by (1); it lacks some of the characteristic properties of the latter, and does not lend itself so readily to algebraic treatment. Its maximum value, to be sure, is 1, found in the case of perfect agreement in rank, when all the numbers D_k are zero. But the minimum value for a given number of entries is not -1. In the case of four individuals, for example, comparison of the arrangements (1, 2, 3, 4) and (4, 3, 2, 1) for $(X_1', \cdots, X_4')$ and $(Y_1', \cdots, Y_4')$ respectively gives $R = -\frac{3}{5}$, and it is readily seen[2] that this is the minimum. Furthermore, the minimum value of R does not necessarily imply complete reversal of order; the same value $-\frac{3}{5}$ is found by comparing (1, 2, 3, 4) with (4, 3, 1, 2) or (3, 4, 1, 2). It is suggested[3] that negative values "large enough to be appreciable" be avoided "by ranking one of the series in the reverse order," with a note that "this reversal of ranks usually modifies the result, but only to an extent that is much smaller than the probable error and therefore negligible." It may be added that a negative result, perhaps too small to be practically significant, may remain even after the reversal. Comparison of (1, 2, 3, 4) with (2, 4, 1, 3) gives $R = -\frac{1}{5}$, and comparison with the inverted arrangement (3, 1, 4, 2) gives $-\frac{1}{5}$ again. The "footrule" has been found useful in practice, but, as Spearman himself emphasizes, its possibilities and limitations are to be recognized by experience, not by inference from what is known about the coefficients r and ρ. The formulas and tables expressing ρ in terms of R depend on an assumption as to the distribution of the data, and are not a matter of universal demonstration.

7. Geometrical interpretation of r. The meaning of the general coefficient r can be made clearer by a geometrical representation. Let the pair of numbers (x_k, y_k) be regarded as the coördinates of a point, with respect to a pair of rectangular axes. Then the n pairs $(x_1, y_1), \cdots, (x_n, y_n)$ are represented by n points distributed somewhere in the plane; these points may be all distinct, or some of them may coincide. They do not all lie on the y-axis, because of the assumption that the x's are not all zero, and it can be said similarly that they do not all lie on the x-axis. Since $\Sigma x_k = \Sigma y_k = 0$, they are so arranged that their center of gravity, if they are regarded as material particles of equal weight, falls at the origin. Apart from these conditions, they may be any n points in the plane. It has been seen that if $r = 1$, there is a positive constant c such that $y_k = cx_k$ for all values of k. This means that the points all lie on the straight line $y = cx$, passing through the origin and running into the first and third quadrants. If $r = -1$, the points lie on a line $y = -cx$, containing the origin

[1] The formula is commonly written with $6\Sigma g$ in place of $3\Sigma|D_k|$, where Σg stands for the sum of the "gains" in rank observed in passing from one arrangement to the other, the "losses" being neglected. As the sum of the "gains" is necessarily equal numerically to the sum of the "losses," and $\Sigma|D_k|$ is the arithmetical sum of all together, $\Sigma|D_k|$ is equal to $2\Sigma g$.

[2] There is no loss of generality in taking $X_1' = 1$, $X_2' = 2$, $X_3' = 3$, $X_4' = 4$, and then the 24 possible arrangements for the numbers Y_k' can be tried out in succession.

[3] Spearman, *loc. cit.*, p. 96, footnote.

and going from the second quadrant into the fourth. If r is neither $+1$ nor -1, they do not lie on any one line.[1] There is a relation between the magnitude of r and the degree of their approach to a linear arrangement. This will be clearest in terms of a still further specialized notation, to be introduced in the following paragraph. It may be observed in passing that if the points (X_k, Y_k) are plotted instead of (x_k, y_k), they will lie on a straight line if $r = \pm 1$, but this line will not generally pass through the origin.

Let σ and τ stand for the quantities

$$\sigma = \sqrt{(\Sigma x_k^2)/n}, \qquad \tau = \sqrt{(\Sigma y_k^2)/n}.$$

These are the *standard deviations* of the x's and of the y's respectively. Let

$$s_k = x_k/\sigma, \qquad t_k = y_k/\tau.$$

It follows from § 3, or directly by substitution of σs_k for x_k and τt_k for y_k in (1), that

$$r = (\Sigma s_k t_k)/\sqrt{(\Sigma s_k^2)(\Sigma t_k^2)}.$$

But $\Sigma s_k^2 = \Sigma t_k^2 = n$, so that the last formula becomes simply

$$r = \frac{1}{n}\Sigma s_k t_k.$$

By what amounts to a repetition of a part of the algebraic work of § 5, it is seen that

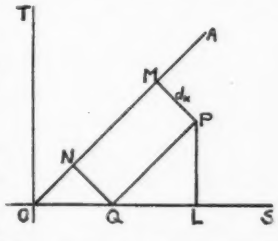
$$r = \frac{1}{2n}\Sigma(2s_k t_k) = \frac{2n - (n - 2\Sigma s_k t_k + n)}{2n} \tag{6}$$

$$= 1 - \frac{\Sigma s_k^2 - 2\Sigma s_k t_k + \Sigma t_k^2}{2n} = 1 - \frac{1}{2n}\Sigma(s_k - t_k)^2.$$

Similarly,

$$r = \frac{-2n + (n + 2\Sigma s_k t_k + n)}{2n} = -1 + \frac{1}{2n}\Sigma(s_k + t_k)^2.$$

As the sums of squares can not be negative, these formulas show[2] once more that r is a number belonging to the interval from -1 to $+1$. They also lend themselves to an immediate geometric interpretation, if the points having the coördinates (s_k, t_k) are plotted with respect to a pair of coördinate axes.



Let P in the figure be an arbitrary one of the points (s_k, t_k), and let OA be the line $t = s$, bisecting the angle between the axes OS and OT. The coördinates of P are $OL = s_k$ and $LP = t_k$. Let the perpendicular distance PM be denoted by d_k. The manner of construction of the rest of the figure will be sufficiently clear without verbal description. It

[1] If they are in a straight line at all, that line contains their center of gravity, and so must pass through the origin.

[2] Cf. Huntington, *loc. cit.*, p. 424.

is seen that

$$d_k = \overline{PM} = \overline{QN} = \overline{OQ}/\sqrt{2},$$
$$\overline{OQ} = \overline{OL} - \overline{QL} = \overline{OL} - \overline{LP} = s_k - t_k,$$
$$d_k = (s_k - t_k)/\sqrt{2}.$$

Other situations of the point P will give a somewhat different figure, but the result will be the same in all cases, except possibly for algebraic sign. So the relation (6) means that

$$r = 1 - \frac{1}{n}\Sigma d_k^2;$$

the coefficient of correlation is less than 1 *by an amount equal to the mean of the squares of the distances of the points* (s_k, t_k) *from the line* $t = s$. The condition $r = 1$, in this notation, means that all the points lie on the 45° line. Similarly, r exceeds -1 by an amount equal to the mean square distance from the other 45° line, $t = -s$.

8. Least squares: the arithmetical mean. The correlation coefficient is found to possess additional meaning in the light of the method of least squares. It is not the purpose of the present article to discuss the theoretical justification of that method, but only to show how its application works out in one or two instances. It is perhaps simplest to begin with an illustration which does not involve the idea of correlation, but is of fundamental importance in itself.

Let $X_1, X_2, \cdots, X_n$ be any n given real numbers, distinct or not. Let it be required to find a number x, so that the quantity $\Sigma(X_k - x)^2$ shall have the smallest possible value. The answer is given by an observation which will be used again in the following section.

The most general expression of the second degree in x, the left-hand member of the general quadratic equation, is $ax^2 + bx + c$. It will be sufficient here to consider the case that $a > 0$. Then the expression can be written in the form

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a};$$

the second member reduces immediately to the first on simplification. One term on the right is independent of x, and the other is positive or zero for all real values of x, since a is positive. So the whole expression takes on its smallest value when the square term is zero, that is, when $x = -b/(2a)$, and the resulting minimum value is

$$\frac{4ac - b^2}{4a} = c - \frac{b^2}{4a}. \tag{7}$$

The quantity to be minimized in the particular problem that was proposed is

$$\sum_{k=1}^{n}(X_k - x)^2 = \sum_{k=1}^{n} X_k^2 - 2x\sum_{k=1}^{n} X_k + nx^2.$$

This can be identified with $ax^2 + bx + c$ by taking $a = n > 0$, $b = -2\Sigma X_k$, $c = \Sigma X_k^2$. The minimizing value of x is $x = -b/(2a) = (\Sigma X_k)/n$; that is, the minimum is given by the arithmetical mean. It will be noticed that this important fact is independent of any restrictive assumption as to the distribution of the numbers X_k.

9. Least squares: lines of regression. To come back to the pairs of observations which have formed the main subject of the paper, let it be required to determine a coefficient λ so that the line $y = \lambda x$ shall give the best possible representation of the arrangement of the points (x_k, y_k), in the sense that the expression $\Sigma(\lambda x_k - y_k)^2$ shall be as small as possible. The quantity to be minimized is $\lambda^2\Sigma x_k^2 - 2\lambda\Sigma x_k y_k + \Sigma y_k^2$, which is in the form $a\lambda^2 + b\lambda + c$, with $a = \Sigma x_k^2 > 0$, $b = -2\Sigma x_k y_k$, $c = \Sigma y_k^2$. The minimum is given by

$$\lambda = -\frac{b}{2a} = \frac{\Sigma x_k y_k}{\Sigma x_k^2} = \frac{\Sigma x_k y_k}{\sqrt{(\Sigma x_k^2)(\Sigma y_k^2)}} \cdot \sqrt{\frac{\Sigma y_k^2}{\Sigma x_k^2}} = r\frac{\tau}{\sigma}.$$

The line $y = \lambda x = r(\tau/\sigma)x$ is called a *line of regression*.[1]

For the sake of simplifying the expression for the minimum value of the sum of squares, as given by (7) of the preceding section, let $p = (1/n)\Sigma x_k y_k$. Then

$$\Sigma x_k^2 = n\sigma^2, \qquad \Sigma y_k^2 = n\tau^2, \qquad \Sigma x_k y_k = np, \qquad r = \frac{p}{\sigma\tau}.$$

Consequently

$$c - \frac{b^2}{4a} = \Sigma y_k^2 - \frac{(\Sigma x_k y_k)^2}{\Sigma x_k^2} = n\tau^2 - \frac{np^2}{\sigma^2} = n\tau^2\left(1 - \frac{p^2}{\sigma^2\tau^2}\right) = n\tau^2(1 - r^2).$$

If the square root of one nth of the minimum sum of squares is denoted by τ_1, then

$$\tau_1^2 = \tau^2(1 - r^2).$$

There is another line of regression, $x = \mu y$, determined by the condition that $\Sigma(\mu y_k - x_k)^2$ shall be a minimum. The value of μ is found to be $r\sigma/\tau$, and σ_1, defined as equal to the square root of one nth of the minimum sum of squares in this case, is given by the equation

$$\sigma_1^2 = \sigma^2(1 - r^2).$$

A broader question, in appearance at least, would be that of determining the best approximating equation of the form $y = \lambda x + \lambda'$, in the sense of the method of least squares. The same method of treatment would show, without much more difficulty, that λ' must be zero, so that the line of regression really gives the solution of this problem as well. The details of the proof need not be given here.[2]

[1] Cf. Huntington, *loc. cit.*, pp. 427–428. The notation is somewhat different.

[2] If λ is given *any* fixed value, say $\lambda = \lambda_0$, the best representation that can be found by varying λ' alone is obtained by taking $\lambda' = 0$; the formal proof depends on the fact that $\Sigma x_k = \Sigma y_k = 0$.

10. Partial correlation. In the simplest case of three variables, a coefficient of partial correlation gives a measure of the degree of dependence of two of the variables on each other, apart from such relationship as arises out of their common dependence on the third. This vague indication is of course only descriptive; its substance will appear from the formal work that follows.

Let $X_1, \cdots, X_n$; $Y_1, \cdots, Y_n$; and $Z_1, \cdots, Z_n$ be three sets of observations on n individuals, and let $x_1, \cdots, x_n$; $y_1, \cdots, y_n$; and $z_1, \cdots, z_n$ be their deviations from their respective means. Let

$$\Sigma x_k^2 = n\sigma^2, \qquad \Sigma y_k^2 = n\tau^2, \qquad \Sigma z_k^2 = n\omega^2,$$

$$r_{12} = \frac{\Sigma x_k y_k}{\sqrt{(\Sigma x_k^2)(\Sigma y_k^2)}}, \qquad r_{13} = \frac{\Sigma x_k z_k}{\sqrt{(\Sigma x_k^2)(\Sigma z_k^2)}}, \qquad r_{23} = \frac{\Sigma y_k z_k}{\sqrt{(\Sigma y_k^2)(\Sigma z_k^2)}},$$

so that σ, τ, ω are the standard deviations of the three sets respectively, and r_{12}, r_{13}, r_{23} are the coefficients of correlation of two of the sets at a time, calculated in the usual way. It is seen at once that

$$\Sigma x_k y_k = n r_{12}\sigma\tau, \qquad \Sigma x_k z_k = n r_{13}\sigma\omega, \qquad \Sigma y_k z_k = n r_{23}\tau\omega.$$

Let μ_1 and μ_2 be the coefficients which minimize the sums of squares $\Sigma(x_k - \mu z_k)^2$ and $\Sigma(y_k - \mu z_k)^2$ respectively. By the results of the preceding section, $\mu_1 = r_{13}\sigma/\omega$, and $\mu_2 = r_{23}\tau/\omega$. Let

$$u_k = x_k - \mu_1 z_k, \qquad v_k = y_k - \mu_2 z_k.$$

The u's and v's may be regarded as what is left of the x's and y's, after the utmost possible has been done to eliminate any (linear) dependence on the z's. At any rate, the definition is to be given in terms of the quantities u_k and v_k, however they may be characterized verbally.

The coefficient of partial correlation between the X's and the Y's is simply the coefficient of correlation between the u's and the v's, defined in the ordinary way:

$$r' = (\Sigma u_k v_k)/\sqrt{(\Sigma u_k^2)(\Sigma v_k^2)}.$$

Its extreme values are therefore $+1$ and -1, as in the case of any other coefficient of correlation.

It is possible to express r' in such a form that its numerical determination does not require the actual calculation of the numbers u_k and v_k. By the preceding section,

$$\Sigma u_k^2 = n\sigma^2(1 - r_{13}^2), \qquad \Sigma v_k^2 = n\tau^2(1 - r_{23}^2).$$

Furthermore,

$$\begin{aligned}\Sigma u_k v_k &= \Sigma(x_k - \mu_1 z_k)(y_k - \mu_2 z_k) \\ &= \Sigma x_k y_k - \mu_1 \Sigma y_k z_k - \mu_2 \Sigma x_k z_k + \mu_1\mu_2 \Sigma z_k^2,\end{aligned}$$

or, by substitution of the equivalents given above for μ_1 and μ_2 and the various sums in the last member,

$$\begin{aligned}\Sigma u_k v_k &= n r_{12}\sigma\tau - r_{13}\frac{\sigma}{\omega}\cdot n r_{23}\tau\omega - r_{23}\frac{\tau}{\omega}\cdot n r_{13}\sigma\omega + r_{13}\frac{\sigma}{\omega}\cdot r_{23}\frac{\tau}{\omega}\cdot n\omega^2 \\ &= n\sigma\tau(r_{12} - r_{13}r_{23}).\end{aligned}$$

Consequently

$$r' = (r_{12} - r_{13}r_{23})/\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}.$$

The more complicated cases of partial correlation follow the same order of ideas, but would require a more elaborate notation for their systematic treatment.

TWO MODELS IN STATISTICAL MECHANICS.

By A. J. LOTKA, Johns Hopkins University.

1. Introduction. Those thermal effects which we commonly observe with our gross senses (assisted on occasion by more or less refined thermometric and other instruments), and which we treat, in terms of such observations, by the method of thermodynamics, are very successfully interpreted, by the method of statistical mechanics, as the observed effects of " concealed motions " of imperceptible particles (molecules). So, to quote only the simplest possible example, the pressure which a gas exerts upon its container is, in this interpretation, recognized as the momentum lost, per unit of time, by the molecules which impinge upon, and are reflected from, the walls of the container.

While the conclusions of statistical mechanics in this and in many other respects stand in excellent harmony with observation and with the thermodynamic setting of physical laws, yet a critical scrutiny reveals certain apparent conflicts which are not resolved without effort. These conflicts are closely bound up with the fact that, in the thermodynamic interpretation, natural processes are essentially unidirectional in time, whereas in mechanics the forward and the backward directions in time are on an equal footing. The motion of a " purely mechanical system " is periodic; for example, one glance at a sine curve, representing the motion of a pendulum, immediately makes it plain to the eye that, cut in two at a crest, the curve is symmetrical to the right and left; motion represented by such a curve is symmetrical as regards $+ t$ and $- t$, when suitable choice of origin is made.[1] A pendulum alone would never enable us to distinguish between yesterday, to-day and to-morrow. But in the processes considered in thermodynamics the case is different. If two bodies in contact in a non-conducting enclosure are observed on three days A, B, C, to show respectively a temperature contrast of 10, 20 and 40 degrees, we know that the day A must be later than B, and B later than C. The progress of heat conduction is *not* indifferent to the sign of t. The physicist commonly expresses this by saying that for any isolated system the *entropy* can only increase. So, for example, when a body M_1 at temperature θ_1 loses a quantity Q of heat by simple *conduction* to another body M_2 at temperature θ_2, the body M_1 is said to lose entropy Q/θ_1 and the body M_2 to gain entropy Q/θ_2. The net gain in entropy $(Q/\theta_2 - Q/\theta_1)$ for the entire system is necessarily positive[2] since θ_1 is necessarily greater than θ_2.

[1] For a rigorous discussion of this point, see Poincaré, *Thermodynamique*, 1908, p. 441.

[2] The entropy of an isolated system increases, not only in the equalization of temperature differences by conduction, but in all *spontaneous* processes, as, for example, in diffusion. A com-

Here, then, thermodynamics and statistical mechanics seem at first sight to be in conflict. The entropy of an isolated system always increases. If the system were composed of a number of particles in mechanical motion, its behavior would be periodic, and its entropy would decrease as often as increase.

The complete analytical examination of the question thus raised we owe chiefly to Boltzmann,[3] who showed that purely mechanical systems do, indeed, in certain circumstances display a property analogous to entropy. He showed, moreover, that the entropy of a gas in a given state stands in a simple relation to the probability of that state, and that the statistical interpretation of the law of increasing entropy amounts to the statement that an isolated system passes always from less probable to more probable states until equilibrium is reached.

A critical understanding of Boltzmann's argument is not reached without considerable effort, and models illustrating on a large scale some of the moot points regarding the behavior of a swarm of molecules may prove a welcome aid. Two such model processes are therefore here submitted.

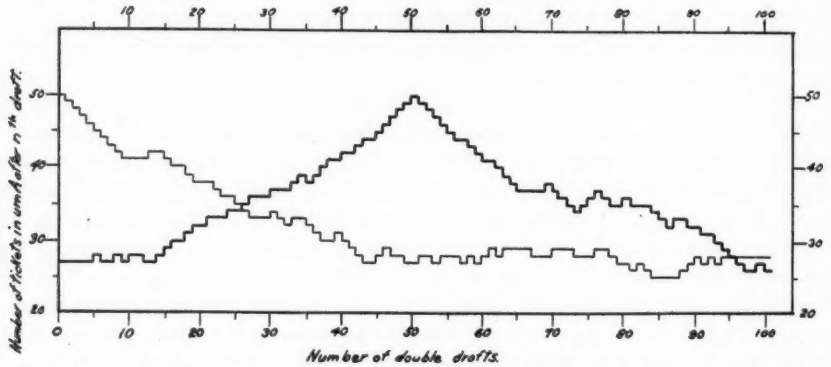


FIG. 1. Staircase curve obtained by plotting history of contents of urn A. Lighter line represents number of black balls in A after n drafts, starting with 50 black balls, and tracing history forward in time. Heavier line represents history of urn A traced forward and *backward* in time from the fiftieth draft.

2. The two models. The first of these two model processes is exemplified in Fig. 1. The more lightly drawn curve in this figure was obtained by drawing blindly balls from two urns A and B, the former containing initially only black balls, the latter only white balls. After each draft the balls drawn were returned to *opposite* urns. This process, which may be regarded as a crude imitation of the mutual diffusion of two gases, was repeated 100 times, and the number of black balls in urn A after each draft was plotted. The curve illustrates in obvious manner the passage of the system from less probable to more probable states, in analogy with the increase in entropy that accompanies the diffusion of two

plete exposition of the significance of entropy cannot be given in any parenthetic statement; for detailed treatment of this matter the reader must be referred to standard works on the subject, such as E. Buckingham's *Theory of Thermodynamics*.

[3] *Vorlesungen über Gastheorie*, 1896, vol. 1, pp. 42 *et seq.*

gases into each other (at constant total volume). The curve further shows small fluctuations, small deviations from strictly even distribution of the black balls in the two urns, persisting indefinitely after sensible equilibrium is reached. This also is in correct analogy with gaseous diffusion. Statistical mechanics admits such small departures, such momentary very slight *decreases* in entropy, when the system is very near equilibrium. Experimentally also these small fluctuations have been demonstrated by Svedberg,[1] Smoluchowski[2] and others. Thermodynamics is silent regarding these minute variations; it does not presume to give information regarding phenomena in the realm of molecular dimensions, but concerns itself only with the grosser *average* manifestations directly observable by the senses.

So far the model process described has only reaffirmed familiar examples,[3] but we have not exhausted its possibilities. We have noted that minor departures from equal distribution of black and white balls between the two urns continue to occur indefinitely. On theoretical grounds we should expect also, on very rare occasions, large departures from the mean, even to the extent of 50 balls starting from equal distribution in the two urns, and gradually assembling, all of them, in one single urn A. Can we hope to convert this theoretical expectation into actuality of observation? Not by frontal attack. Urns and balls would likely be worn to dust before that miraculous draft occurred. But our model will yield to persuasion. A simple artifice will give us a view of that lonely peak rising to the level of 50 from among a long stretch of small fluctuations.

Improbable things are happening all around us, but for the most part they are of no practical significance, and we fail to notice them. The miraculous draft which assembles in one urn 50 balls initially spread evenly in the two urns is happening under our eyes, but we fail to distinguish the balls, except as to *blackness and whiteness*. Let us slightly recast our model. Let us number the balls, or, since this is more convenient, use tickets numbered from 1 to 50 for the initial charge of urn A, and fifty more tickets, numbered from 51 to 100, for the initial charge of urn B. We now make a series of drafts as before, but this time we keep records of all the tickets by number. In an actual experiment 50 such drafts were made. At the end of the series we noted the contents of each urn, either by direct inspection or by consulting the record of tickets drawn. Starting with the urn contents just as they were at the end of the first series, we then made a second series of 50 drafts. The identity of all the tickets in urn A at the beginning of the second series being known, their previous as well as their subsequent history could be traced from the records. The history thus revealed in the experiment here recorded is shown in the more heavily drawn curve of Fig. 1. As will be seen, this contains a peak reaching up to the extreme possible limit of 50 tickets. It might be thought that this does not represent a rare event, inasmuch as we can produce at will as many of these peaks as we please. But

[1] Th. Svedberg, *Die Existenz der Moleküle*, Leipzig, 1912, p. 148.
[2] M. v. Smoluchowski, *Bull. Acad. Cracovie*, 1916, p. 218.
[3] Compare P and T. Ehrenfest, *Physikalische Zeitschrift*, 1907, vol. 8, p. 311.

any skeptic who has the patience can convince himself that such peaks are truly rare, if he will run two series, as described, the first of one million, and the second of another million drafts. It is safe to predict that he will encounter but one peak running up to 50. Nevertheless, the model makes it quite clear that such peaks are not only possible, but do actually occur. In point of fact it can be shown that, starting with black and white balls evenly distributed, 25 of each in each urn, we may expect to see all black balls in one urn about once in $3\frac{1}{2}$ million years, allowing one draft every second.

This illustrates very well how the apparent conflict between the verdict of thermodynamics and that of statistical mechanics can be harmonized. Thermodynamics says, two gases from two connected containers will mix completely, and will stay mixed forever. Statistical mechanics says, the gases will first mix, and then in N years they will unmix again, but N is a number large beyond all human comprehension, and for all practical purposes infinite.

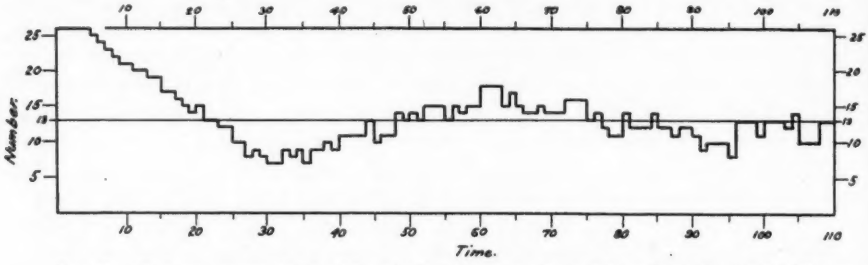


FIG. 2. Staircase curve obtained by plotting number of pendulums (out of a total of 26) on left of median line at successive epochs. The unit of the time scale is one tenth of a second.

As has already been indicated, the systems treated in statistical mechanics are, strictly speaking, periodic. The question accordingly arises how their periodic motion can exhibit phenomena analogous to increase in entropy. The second model illustrates this point very clearly. Twenty-six pendulums of periods $T = .5, .6, .7, .8, \cdots 2.9, 3.0$ seconds are started simultaneously from the median position to the left, and are then allowed to oscillate undisturbed. Count is made, at the end of every tenth of a second, of the number of pendulums on the left of the median. In this way the staircase curve, Fig. 2, was obtained (in this case by computation, not by observation). It will be seen that in the fragment of a period covered by the record, this exhibits all the characteristics of a "passage from a less probable to a more probable distribution," though, in point of fact, we know that the system has a perfectly definite period of about 7385 years, and moves in an absolutely determinate manner. The appearance of "chance" in this perfectly determinate mechanical process is brought into still greater prominence if we plot the deviations, from the mean, of the number of pendulums found on the left of the median lines, at successive counts. We thus[1] obtain the points

[1] Discarding the first 13 out of 425 counts, as being obviously unusual and strongly influenced by initial conditions.

indicated by small circles in Fig. 3. The polygonal diagram in Fig. 4 represents the corresponding frequencies for the simpler system of four pendulums, with the periods $T = .5, .6, .7, .8$ seconds; these are found to lie, very nearly indeed, on points corresponding to the coefficients of the expansion of $(1 + 1)^4$. A little reflection shows that if this relation holds for one series of pendulums of the kind here considered, it should also hold when one more pendulum is added to the series, and so on. We may test this conclusion for the case of 26 pendulums. Instead of plotting the coefficients of $(1 + 1)^{26}$, however, it is more convenient, and practically equivalent, to plot the Gaussian error curve with a standard deviation of $\sqrt{26/4}$ ($= 2.549$). This has been done in Fig. 3 and, as will be seen, the fit is good, considering the smallness of the sample (412 observations, extending over 41.2 seconds out of a total period of 7385 years).

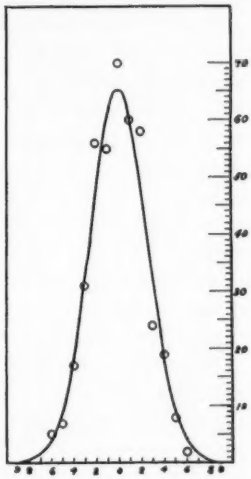


FIG. 3. Frequency of deviations from mean in number of pendulums (out of a total of 26) to left of median, in the case represented by Fig. 2.

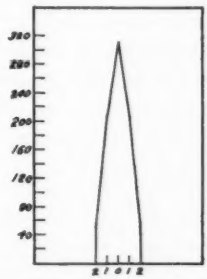


FIG. 4. Frequency of deviations from mean, as in Fig. 3, but plotted for 4 pendulums making 840 counts to cover one entire period

What do our model curves teach us regarding the alleged asymmetry of time? It is at once obvious that in the model of Fig. 1 the heavy curve makes no essential distinction between the forward and the backward direction in time. We know that high peaks are very rare, and therefore, if we find the system in a state corresponding to a high point of the curve, we know that we are very probably near the crest of a peak, so that we are either actually descending, or very soon shall be. But all this is equally true whether we read the curve from left to right or from right to left. Similar remarks apply to Fig. 2, the details of which may be left to the reader to work out. The truly significant fact is that high peaks occupy only a vanishingly small portion of the total base line, so that if the system is in a very improbable state, it is extremely likely that it will very soon pass into a much

more probable one—but also, it is extremely likely that it was in a very much more probable state a short time ago. The proposition holds in either sense, making no distinction between $+ t$ and $- t$. Whence then comes our intuitive, subjective sense of the asymmetry of time? Perhaps it is in some way connected with the integration constants; for it is only the *differential* equations that do not respect the sign of t. Or perhaps the fundamental directedness of the world events is not expressed in the equations of ordinary dynamics, but strikes its roots deeper into the underlying quantum mechanics.[1] As yet we stand, here, upon uncertain ground of speculation.

In conclusion attention may be drawn to a psychological and biological significance of the models here presented. It appears at first sight as if there were a fundamental difference in character between the first and the second model, since it is essential for the operation of the urn model that the drawing be done *blindly*, so as to give *chance* a part in the process; whereas the pendulum model we operate with our eyes open, apparently in full consciousness of what is going on. Chance seems to play no part here, the system is mechanically determinate.

But there is a blindness which is not of the eye, and there is a vision that surpasses optical vision. The same struggle for existence which has developed in man the organ of sight, to depict for him the external world, to furnish him with a map on which to base his plan of campaign, is also developing his internal vision, whereby he extends his world-picture beyond the powers of the bodily eye. Whether I peep into the urn and manipulate the drafts by the light of my eyes; or whether, in the light of my knowledge of mechanics, I adjust the pendulums to equal lengths and phases; or again, whether, in the more serious affairs of life, I employ these same faculties to diverse ends, the effect is the same. In greater measure or less these organs and faculties emancipate me from the bonds of the fortuitous and make me, in this sense, a controller of events. Their function is to substitute choice for chance, to introduce aimed collisions in place of random encounters.

THE NUMBERS OF REPRESENTATIONS OF INTEGERS IN CERTAIN FORMS $ax^2 + by^2 + cz^2$.

By E. T. BELL, University of Washington.

1. Introduction. The number $N(n = ax^2 + by^2 + cz^2)$ of sets (x, y, z) of integers $x, y, z \gtreqless 0$ satisfying $n = ax^2 + by^2 + cz^2$ is called the number of representations of n in the form $ax^2 + by^2 + cz^2$. When $a = b = c = 1$ this number was determined by Gauss.[2]

[1] *Cf.* G. Breit. Are Quanta Unidirectional? *Phys. Rev.*, vol. 22, 1923, p. 313.

[2] *Disquisitiones Arithmeticæ*, Arts. 292, 293 (Leipzig, 1801). *Cf.* Dickson, *History of the Theory of Numbers*, vol. 2, Preface, pp. ix, x.

The only other specific results for $N(n = ax^2 + by^2 + cz^2)$ appear to be an unproved assertion of Liouville (1869) for the case $(a, b, c) = (1, 2, 3)$, and incomplete results for $(a, b, c) = (1, 1, 2)$ (Torelli, 1878), and $(a, b, c) = (1, 2, 2)$ (Stieltjes, 1883).[1]

Assuming Gauss' theorem and the classic $N(n = x^2 + y^2) = 4\xi(m)$, where $n = 2^\alpha m$, $\alpha \geqq 0$, m odd, $\xi(m) =$ the number of $4k + 1$ divisors of m minus the number of $4k + 3$ divisors, we shall obtain *complete* enumerations

$$N\,(n = ax^2 + by^2 + cz^2)$$

for each of the nine forms

$$\begin{array}{lll} x^2 + y^2 + 2z^2, & x^2 + 4y^2 + 8z^2, & x^2 + 8y^2 + 8z^2, \\ x^2 + 2y^2 + 2z^2, & x^2 + y^2 + 8z^2, & x^2 + 4y^2 + 4z^2, \\ x^2 + 2y^2 + 4z^2, & x^2 + 2y^2 + 8z^2, & x^2 + y^2 + 4z^2, \end{array}$$

which, it is easily seen, with Gauss' $x^2 + y^2 + z^2$, include all possible cases of $N(n = ax^2 + by^2 + cz^2)$ in which each of a, b, c is a power, not higher than the third, of 2. The type of theorem indicated in § 14 is of particular interest.

2. The Theorem of Gauss. The new results of this paper will be more interesting to any who have not specialized in arithmetic if we first describe in a general way the nature of Gauss' theorem, which is not at all obvious.

The totality of binary quadratic forms $ax^2 + 2bxy + cy^2$ with integer coefficients a, b, c having their determinants $b^2 - ac$ equal to a constant $D < 0$ is segregated into classes, all those forms, called (properly) equivalent, which are such that any one can be derived from any other by a linear substitution

$$x' = \alpha x + \beta y, \qquad y' = \gamma x + \delta y, \qquad \text{where} \qquad \alpha\delta - \beta\gamma = 1,$$

and $\alpha, \beta, \gamma, \delta$ are integers, being put into the same class. The number of classes for each D is finite. For any D there exists a finite number of forms, called reduced, whose coefficients satisfy certain inequalities, and in each class there is at least one reduced form, which evidently is equivalent to each of the forms in its class and can therefore be taken as the representative of all. If the outer coefficients of the representative form are not both even, the corresponding class is called odd, otherwise even. The numbers of odd, even classes for the negative determinant $-n$ are written $F(n)$, $F_1(n)$, and $E(n) \equiv F(n) - F_1(n)$. Either from arithmetic or elliptic series it is proved that

$$\begin{array}{lll} 3E(8n + 3) = 2F(8n + 3), & E(8n + 7) = 0, & E(4n) = E(n), \\ E(4n + 1) = F(4n + 1), & E(8n + 2) = F(4n + 2), & F(4n) = 2F(n). \end{array}$$

By convention a class equivalent to $a(x^2 + y^2)$ contributes $\frac{1}{2}$ to F or to F_1; one equivalent to $a(2x^2 + 2xy + 2y^2)$ contributes $\frac{1}{3}$ to F_1; $F(0) = 0$, $E(0) = \frac{1}{12}$.

The theorem of Gauss states that

$$N(n = x^2 + y^2 + z^2) = 12E(n). \tag{1}$$

[1] For references and summaries *cf.* Dickson, *loc. cit.*, vol. 2, pp. 294, 295; vol. 3, pp. 133, 216.

3. Notation, Series. Henceforth $m_i\,(i = 1, 2, \cdots)$ denote odd integers $\geqq 0$, $n_i\,(i = 1, 2, \cdots)$ even or odd integers $\gtrless 0$; $n \geqq 0$ is even or odd, $m > 0$ is odd, $\alpha \geqq 0$ is even or odd. When m, n, m_i, n_i occur under a Σ, the summation is with respect to all values of the m, n, m_i, n_i consistent with the above definitions. The functions $\vartheta_\alpha(q)$ are defined by

$$\vartheta_3(q) = \Sigma q^{n_i^2}, \qquad \vartheta_2(q) = \Sigma q^{m_i^2/4}, \qquad \vartheta_0(q) = \vartheta_3(-q).$$

From these definitions and the theorem on $N(n = x^2 + y^2)$ quoted in § 1 it is easy by comparing coefficients of like powers of q to verify

$$\vartheta_0(q)\vartheta_3(q) = \vartheta_0^2(q^2), \quad (2) \qquad \vartheta_3^2(q^2) - \vartheta_0^2(q^2) = 2\vartheta_2^2(q^4), \quad (5)$$

$$2\vartheta_2(q^8)\vartheta_3(q^8) = \vartheta_2^2(q^4), \quad (3) \qquad \vartheta_3^2(q) + \vartheta_0^2(q) = 2\vartheta_3^2(q^2), \quad (6)$$

$$\vartheta_3^2(q^4) - \vartheta_2^2(q^4) = \vartheta_0^2(q^2), \quad (4) \qquad \vartheta_3^2(q^4) + \vartheta_2^2(q^4) = \vartheta_3^2(q^2), \quad (7)$$

$$2e^{i\pi/4}\vartheta_0(q^8)\vartheta_2(q^8) = \vartheta_2^2(iq^4), \quad (i = \sqrt{-1}); \quad (7.1)$$

or, if preferred, (2) – (7.1) can be taken as well known from elliptic functions.

Now obviously the algebraic equivalent of (1) is

$$\vartheta_3^3(q) = 12\Sigma q^n E(n). \quad (8)$$

If in (8) q be replaced by $-q$, there follow by elementary algebra[1] the important identities of Kronecker and Hermite:

$$\vartheta_0(q^4)\vartheta_2^2(q^4) = 4\Sigma q^{4n+2}(-1)^n F(4n+2), \quad (9)$$

$$\vartheta_0^2(q^4)\vartheta_2(q^4) = 4\Sigma q^{4n+1}(-1)^n F(4n+1), \quad (10)$$

$$\vartheta_2(q^4)\vartheta_3^2(q^4) = 4\Sigma q^{4n+1}F(4n+1), \quad (11)$$

$$\vartheta_2^2(q^2)\vartheta_3(q^2) = 4\Sigma q^{2n+1}F(4n+2), \quad (12)$$

$$\vartheta_2^3(q^4) = 8\Sigma q^{8n+3}F(8n+3). \quad (13)$$

As all results concerning the nine forms to be discussed are immediate consequences of (2) – (13), it follows that for these forms the complete enumeration is implicit in the classical theorems for 2 and 3 squares. It will not be necessary to preserve all the elementary algebraic details for each case, but the reader who cares to supply them will find much more information concerning the forms than that recorded here. I have verified the results numerically.

4. $N(n = x^2 + y^2 + 2z^2) \equiv N(n)$. Multiply (7) by $\vartheta_3(q^2)$, and use (8), (12) on the right of the resulting identity:

$$\Sigma q^{n_1^2+n_2^2+2n_3^2} = 4\Sigma q^{2n+1}F(4n+2) + 12\Sigma q^{2n}E(n).$$

Equating coefficients of like powers of q, we find

$$N(2n+1) = 4F(4n+2), \qquad N(2n) = 12E(n).$$

[1] The extremely simple details are given in full in a note to appear in the *Bulletin of the American Mathematical Society*.

5. Reduction Formulas. Evidently

$$N(pn_1 = pax^2 + pby^2 + pcz^2) = N(n_1 = ax^2 + by^2 + cz^2).$$

Let p be prime, and n_1 prime to p. Every integer n is of the form $p^\alpha n_1$. In order that $p^{\alpha+1}n_1 = x^2 + pby^2 + pcz^2$ shall have solutions it is necessary that $x \equiv 0 \bmod p$; say $x = px_1$. Hence

$$N(p^{\alpha+1}n_1 = x^2 + pby^2 + pcz^2) = N(p^\alpha n_1 = px_1^2 + by^2 + cz^2),$$

and clearly the suffix may be dropped from x_1. Obviously

$$N(n = ax^2 + by^2 + cz^2)$$

is invariant for all permutations of x, y, z. We shall rearrange x, y, z when necessary so that $a \leqq b \leqq c$.

In the present discussion $p = 2$, so that $n_1 = m$. As an example of the reduction,

$$N(2^{\alpha+1}m = x^2 + 2y^2 + 2z^2) = N(2^\alpha m = x^2 + y^2 + 2z^2).$$

When $N(m = x^2 + 2y^2 + 2z^2)$ is known, the further evaluation of

$$N(n = x^2 + 2y^2 + 2z^2)$$

is referred by the above reduction to § 4.

6. $N(n = x^2 + 2y^2 + 2z^2) \equiv N(n)$. Since $m_i^2 \equiv 1 \bmod 8$, $4n_i^2 \equiv 0$ or $4 \bmod 8$, we have $2(x^2 + y^2) \equiv 0, 2, 4 \bmod 8$, and therefore

$$\begin{aligned}
N(8n + 1 = x^2 + 2y^2 + 2z^2) &= N(8n + 1 = m_1^2 + 8n_2^2 + 8n_3^2),\\
N(8n + 5 = x^2 + 2y^2 + 2z^2) &= N(8n + 5 = m_1^2 + 2m_2^2 + 2m_3^2),\\
N(8n + 3 = x^2 + 2y^2 + 2z^2) &= 2N(8n + 3 = m_1^2 + 2m_2^2 + 8n_3^2).
\end{aligned}$$

Now $N(8n + 1)$ is the coefficient of q^{8n+1} in $\Sigma q^{m_i^2} \times \Sigma q^{8n_2^2+8n_3^2}$; $N(8n + 5)$ is the coefficient of q^{8n+5} in $\Sigma q^{m_1^2} \times \Sigma q^{2m_2^2+2m_3^2}$. These direct us to

$$\vartheta_0(q^2)\vartheta_3(q^2)\vartheta_2(q^4) = \vartheta_0^2(q^4)\vartheta_2(q^4),$$

written down from (2), and as in § 4 this gives

$$\begin{aligned}
N(4n + 1 = m_1^2 + 2n_2^2 + 8n_3^2)&\\
- N(4n + 1 = m_1^2 + 2m_2^2 + 2n_3^2) &= 4(-1)^n F(4n + 1).
\end{aligned}$$

Separating cases of n even or odd we get $N(8n + 1)$, $N(8n + 5)$, which combine into the result for $N(4n + 1)$ stated below. Similarly (3) multiplied by $\vartheta_2(q^4)$ gives the value of $N(8n + 3)$. The rest follow immediately by applying § 5 to § 4, and we have finally

$$\begin{aligned}
N(4n + 1) &= 4F(4n + 1), \qquad N(4n + 2) = 4F(4n + 2),\\
N(8n + 3) &= 8F(8n + 3), \qquad N(8n + 7) = 0,\\
N(2^{\alpha+2}m) &= 12E(2^\alpha m).
\end{aligned}$$

7. $N(n = x^2 + 2y^2 + 4z^2) \equiv N(n)$. The first of the following comes from the identity obtained by multiplying (7.1) throughout by $\vartheta_0(iq^4)$, the rest by applying § 5 to § 6:

$$N(2n+1) = 2F(4n+2), \qquad N(8n+2) = 4F(4n+1),$$
$$N(8n+4) = 4F(4n+2), \qquad N(16n+6) = 8F(8n+3),$$
$$N(16n+14) = 0, \qquad N(2^{\alpha+3}m) = 12E(2^\alpha m).$$

8. $N(n = x^2 + 4y^2 + 8z^2) \equiv N(n)$. From (4) multiplied throughout by $\vartheta_0(q^4)$ we get the first of the following; the formulas for even numbers come from § 5 applied to § 4:

$$N(4n+1) = 2F(8n+2), \qquad N(4n+2) = N(4n+3) = 0,$$
$$N(8n+4) = 4F(4n+2), \qquad N(2^{\alpha+3}m) = 12E(2^\alpha m).$$

9. $N(n = x^2 + y^2 + 8z^2) \equiv N(n)$. If $2^{\alpha+1}m$ is represented in this form, x, y are both even or both odd. The case in which they are both even is reduced by § 5 to § 4. Of the rest below, $N(4n+1)$ is from (3) multiplied throughout by $\vartheta_0(q^4)$, $N(8n+2)$ is from (3) multiplied by $\vartheta_3(q^8)$.

$$N(4n+1) = 4F(8n+2), \qquad N(4n+3) = N(8n+6) = 0,$$
$$N(8n+2) = 8F(4n+1), \qquad N(8n+4) = 4F(4n+2),$$
$$N(2^{\alpha+3}m) = 12E(2^\alpha m).$$

10. $N(n = x^2 + 2y^2 + 8z^2) \equiv N(n)$. The second of the following is from the identity obtained by multiplying (3) by $\vartheta_2(q^4)$; the first is from the first in § 11; those for even numbers come from applying § 5 to § 7.

$$N(8n+1) = 4F(8n+1), \qquad N(8n+3) = 4F(8n+3),$$
$$N(8n+5,7) = 0, \qquad N(4n+2) = 2F(4n+2),$$
$$N(16n+4) = 4F(4n+1), \qquad N(16n+8) = 4F(4n+2),$$
$$N(32n+12) = 8F(8n+3), \qquad N(32n+28) = 0,$$
$$N(2^{\alpha+4}m) = 12E(2^\alpha m).$$

11. $N(n = x^2 + 8y^2 + 8z^2) \equiv N(n)$. The first comes from (2) multiplied by $\vartheta_2(q^2)$; to get the formulas for even numbers apply § 5 to § 6.

$$N(8n+1) = 4F(8n+1), \qquad N(8n+3,5,7) = N(4n+2) = 0,$$
$$N(16n+4) = 4F(4n+1), \qquad N(16n+8) = 4F(4n+2),$$
$$N(32n+12) = 8F(8n+3), \qquad N(32n+28) = 0,$$
$$N(2^{\alpha+4}m) = 12E(2^\alpha m).$$

12. $N(n = x^2 + 4y^2 + 4z^2) \equiv N(n)$. The enumerations for this form and the next are read off at once from (9)–(13) and by using § 5 on (1).

$$N(4n) = 12E(n), \qquad N(4n+1) = 4F(4n+1),$$
$$N(4n+2) = N(4n+3) = 0.$$

13. $N(n = x^2 + y^2 + 4z^2) \equiv N(n)$. As indicated we find

$$N(4n) = 12E(n), \qquad N(4n+1) = 8F(4n+1),$$
$$N(4n+2) = 4F(4n+2), \qquad N(4n+3) = 0.$$

14. Case when the number to be represented is a square. Many of the foregoing theorems can be remarkably simplified when the number to be represented is a square or a power of two times a square. In a paper to appear elsewhere I have shown that if $n = 2^{\alpha}m$, then $F(n^2) = 2^{\alpha-1}S(m)$, $E(n^2) = \frac{1}{2}S(m)$, where, if $m = \Pi p_i^{\beta_i}$ is the prime factor resolution of m,

$$S(m) = \Pi[p_i^{\beta_i} + \{1 - (-1)^{\frac{1}{2}(p_i-1)}\}\zeta_1(p_i^{\beta_i-1})],$$

and $\zeta_1(n)$ = the sum of all the divisors of n.

Hence, for example, on referring to § 11, we have

$$N(2^{2\alpha+4}m^2 = x^2 + 8y^2 + 8z^2) = 12E(2^{2\alpha}m^2) = 6S(m),$$

expressing $N(2^{2\alpha+4}m^2)$ without class number functions. The remaining theorems of this kind implicit in §§ 4–13 can be read off by inspection.

CONICAL LOCI ASSOCIATED WITH THE MOTION OF A RIGID BODY ABOUT A POINT.

By E. L. REES, University of Kentucky.

1. The writer gave recently in the MONTHLY (*1923*, 290–296) a vector treatment of the motion of a rigid body in a plane. In the present paper the theory relating to the conical loci associated with the motion of a rigid body with one point fixed is treated by the same vector methods, the advantages of which are more strikingly illustrated here than in the previous article.[1]

2. Instantaneous Axis of Rotation. The following fundamental theorem may be proved vectorially in a number of ways.[2]

The motion of a rigid body one point of which is fixed is at each instant a rotation about an instantaneous axis passing through the fixed point.

3. Polar Cones. The instantaneous axis in the course of its motion generates a cone in space and a cone in the body called the *polar cones*. We shall call the space locus of the inst. axis and its body locus the *space cone* and the *body cone* respectively.

Let $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ be the axes fixed in the body with origin at the fixed point. The angular velocity vector $\boldsymbol{w}$ when referred to the moving trihedral is given by $\boldsymbol{w}' = \boldsymbol{w}\cdot\boldsymbol{i}'\boldsymbol{i}' + \boldsymbol{w}\cdot\boldsymbol{j}'\boldsymbol{j}' + \boldsymbol{w}\cdot\boldsymbol{k}'\boldsymbol{k}' = \Sigma\boldsymbol{w}\cdot\boldsymbol{i}'\boldsymbol{i}'$.

Differentiating with respect to t, we get

$$\dot{\boldsymbol{w}}' = \Sigma\dot{\boldsymbol{w}}\cdot\boldsymbol{i}'\boldsymbol{i}' + \Sigma\boldsymbol{w}\cdot\dot{\boldsymbol{i}}'\boldsymbol{i}'.$$

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[1] A synthetic treatment of the subject of this paper is given in Schoenflies' "*Geometrie du Mouvement*," Paris, 1893, pp. 52–81.

For a vector treatment (Burali-Forti notation) of velocities and accelerations of points of a rigid body the reader is referred to an article by Ziwet and Field in the MONTHLY (*1916*, 371–381).

[2] See this MONTHLY (*1918*, 127) for a vector proof of this theorem; see also Gibbs-Wilson "*Vector Analysis*," pp. 131–132.

Now all of the terms of the second summation vanish since $i' = w \times i'$, etc. Therefore $\dot{w}' = \dot{w}$, which shows that the cones are tangent to each other along the inst. axis, since the tangent plane of each is determined by w and $\dot{w}$, and that the angles of corresponding pairs of consecutive elements are equal. Hence the theorem:

The body cone rolls without slipping on the space cone.

It is obvious that in general the elements of the body cone are the only lines of the body whose trajectory cones have cuspidal lines (lines instantaneously stationary) and that the locus of these cuspidal lines is the space cone.

4. Cone of Inflections. Each line of the body through the fixed point generates a cone. We shall now find the instantaneous locus of those lines which are lines of inflection of their trajectory cones. These lines are determined by the vectors p (position vectors of the points of the body) which satisfy the condition $[p\dot{p}\ddot{p}] = 0$. Since $\dot{p} = w \times p$ and $\ddot{p} = \dot{w} \times p + w \times (w \times p)$, this condition leads to the equation

$$[w\dot{w}p]p^2 - (w \cdot p)^3 + w^2 w \cdot pp^2 = 0,$$

which represents a cone of the third order called the *cone of inflections*. Regarding $|p|$ as constant, and differentiating this equation with respect to u, a parameter which fixes the position of p on the cone, we have

$$[w\dot{w}p_u]p^2 - 3(w \cdot p)^2(w \cdot p_u) + w^2 p^2 w \cdot p_u = 0.$$

If u has the value for which the element, determined by p, coincides with the inst. axis, then $w \cdot p_u = 0$, and it results that $[w\dot{w}p_u] = 0$. Consequently *the cone of inflections is tangent to the polar cones along the inst. axis.*

5. Cone of Cusps. Each plane of the body through the fixed point in the course of its motion envelops a cone. Let us now find the locus of the characteristics of the planes through the fixed point which are cuspidal lines of these envelopes.

The characteristic of a plane $a \cdot r = 0$, where a is normal to the plane fixed in the body and of unit length, is the intersection of this plane and the plane $\dot{a} \cdot r = 0$. It thus passes through the fixed point and has the direction of $a \times \dot{a}$. To be a cuspidal line this characteristic must be instantaneously stationary, the condition for which is

$$(a \times \dot{a}) \times (a \times \dot{a})_t = (a \times \dot{a}) \times (a \times \ddot{a}) = 0.$$

But this is equivalent to $[a\dot{a}\ddot{a}] = 0$. Hence *the normals to the planes which are tangent to their envelopes along cuspidal lines are the elements of the cone of inflections.* The envelope of these cuspidal tangent planes, at a given instant, is a cone.[1] This cone and the cone of inflections are related as follows: each cone is the envelope of the planes perpendicular to the other, and each is the locus of the lines through the fixed point normal to the other.

[1] This cone is called by some authors the cone of cusps, but it seems more appropriate for our purposes to use this name for the conical locus of the cuspidal lines of the plane envelopes as indicated below.

As just seen, the characteristics of the planes, which are cuspidal lines of the envelopes, are determined by the vectors $a \times \dot{a}$, where a is subject to the condition $[a\dot{a}\ddot{a}] = 0$. Let $q = \mu a \times \dot{a}$. By resolving w along a and q we obtain

$$a = \frac{w - w \cdot q^{-1} q}{w \cdot a}.$$

This value of a, when substituted in

$$[a\dot{a}\ddot{a}] = q \cdot (\dot{w} \times a + w \times (w \times a)) = 0,$$

gives $$[wq(\dot{w} + w \cdot q^{-1} w \times q)] = 0,$$

or $$[w\dot{w}q]q^2 + (w \cdot q)^3 - w^2 w \cdot qq^2 = 0.$$

Thus the locus of the cuspidal lines is a cone of the third order. We shall call this cone the *cone of cusps*.[1]

On differentiating the last equation with respect to u and letting q have the direction of the inst. axis (note that the inst. axis is an element of the cone of cusps) it results that $[w\dot{w}q_u] = 0$, which shows that *the cone of cusps is tangent to the polar cones along the inst. axis.*

The cone of cusps and the cone of inflections are symmetric to each other with respect to their common tangent plane which is also the common tangent plane of the polar cones; for, their equations show that to each element of either cone there corresponds an element of the other which makes the same angle with the inst. axis and the supplementary angle with the vector $w \times \dot{w}$. Thus the elements of the two cones are associated in symmetric pairs.

The polar axis (see next Art.) of an element, determined by p, is normal to this element, since $[p\dot{p}\ddot{p}] = 0$, and has the direction of $p \times \dot{p}$. We shall now show that *the locus of the polar axes of the elements of the cone of inflections is the cone of cusps.* Substituting $q = \mu p \times \dot{p}$ in the equation of the cone of cusps, we find that the resulting equation is exactly the equation in p of the cone of inflections, so that if p is an element of the cone of inflections, q is an element of the cone of cusps as was to be proved. We may say then that *the corresponding elements of the cone of inflections and the cone of cusps are perpendicular to each other and are coplanar with the inst. axis* (line of tangency of the cones).

6. Orthogonal Cone. The equation of the normal plane of the trajectory of a point P is $(r - p) \cdot \dot{p} = r \cdot \dot{p} = 0$. It thus passes through the fixed point. The polar axis of this trajectory is the intersection of $r \cdot \dot{p} = 0$ and $r \cdot \ddot{p} = 0$, and therefore passes through the fixed point and has the direction of $\dot{p} \times \ddot{p}$. But since $\dot{p} = w \times p$ and $\ddot{p} = \dot{w} \times p + w \times (w \times p)$, the direction of $\dot{p} \times \ddot{p}$ is independent of the magnitude of p. Therefore the polar axis is the same for the trajectories of all points of a line through the fixed point and is called the *polar axis* of this line.

We now proceed to find the locus of lines whose polar axes are perpendicular

[1] In the inverse motion the cone of inflections and the cone of cusps interchange rôles. See Schoenflies' p. 74.

to the inst. axis. The condition here is $[w\dot{p}\ddot{p}] = 0$. Consequently,

$$[ww \times p\dot{w} \times p + w \times (w \times p)] = 0$$

is the equation of our locus which is therefore a *quadratic cone*. Substituting $\dot{w} = \dot{w}w_1 + w\dot{w}_1$ in this equation, we get $[ww \times p\dot{w}_1 \times p - wp] = 0$, which, assuming $w \neq 0$, is equivalent to $w \times p \cdot [(\dot{w}_1 \times w_1 + w) \times p] = 0$. *This cone is therefore the locus of the intersection of perpendicular planes through the inst. axis and the line determined by* $\dot{w}_1 \times w_1 + w$. From the manner of its generation it is called an *orthogonal cone*.

The last equation may be written in the form $(p \times \dot{p}) \cdot (\dot{w}_1 \times w_1 + w) = 0$, from which follows the theorem:

The tangent planes of the trajectory cones of the elements of the orthogonal cone form a pencil which has for its axis the element (determined by the vector $\dot{w}_1 \times w_1 + w$) *of the orthogonal cone diametrically opposite the inst. axis.*

The plane of w and $\dot{w}_1 \times w_1 + w$, which is a diametral plane of the orthogonal cone, is normal to the common tangent plane of the polar cones, and *the orthogonal cone is tangent to the polar cones along the inst. axis.* This cone is of course also tangent to the cone of inflections and to the cone of cusps along their line of tangency.

7. Cone of Lines with Stationary Polar Axes. If the polar axis of a line, determined by p, is instantaneously stationary, we have

$$(\dot{p} \times \ddot{p}) \times (\dot{p} \times \ddot{p})_t \equiv (\dot{p} \times \ddot{p}) \times (\dot{p} \times \dddot{p}) = 0,$$

which is equivalent to $[\dot{p}\ddot{p}\dddot{p}] = 0$. Since the derivatives of p are linearly homogeneous in p, this equation represents a *cone of the third order*, which is *the conical locus of lines with stationary polar axes.* Since the polar axes for points of this cone are stationary the osculating circles of the trajectories of these points are also stationary, or in other words, the cone of lines with stationary polar axes is the locus of points for which the osculating circles have contact of the third order with the trajectories. This cone is also the locus of points at which the osculating planes of their trajectories are momentarily stationary.

8. Cone of Stationary Polar Axes. The polar axis corresponding to p is determined by $q = \dot{p} \times \ddot{p} = [w\dot{w}p]p + (w \times p)^2 w$. Multiplying by $w \times \dot{w} \cdot$ and $w \times$ respectively, we get $[w\dot{w}q] = [w\dot{w}p]^2$ and $w \times q = [w\dot{w}p]w \times p$, so that

$$p = \frac{(w \times q) \times w^{-1}}{\pm \sqrt{[w\dot{w}q]}} + \lambda w.$$

Substituting this expression for p in the equation $q = \dot{p} \times \ddot{p}$, we find

$$\lambda = \frac{[w\dot{w}q]w^{-1} \cdot q - (w \times q)^2}{\pm \sqrt{[w\dot{w}q]^3}},$$

so we may express p in terms of q as follows:

$$p = \frac{[w\dot{w}q]q - (w \times q)^2 w}{\pm \sqrt{[w\dot{w}q]^3}}.$$

Substituting this final expression for p in the equation of the cone of lines with stationary polar axes we obtain, after simplifying, an equation of the third degree in q. Hence *the locus of stationary polar axes is a cone of the third order.*

QUESTIONS AND DISCUSSIONS.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS.

I. A PROPERTY OF THE ISOGONAL CENTERS OF A TRIANGLE.

By H. M. LUFKIN, Dunkirk, N. Y.

The following is an interesting property of the isogonal centers of a triangle which I believe is new.

It is known that the I_+ point (*cf.* my paper in this MONTHLY, *1923*, 127–131) for any triangle having an angle greater than 120° is outside the triangle, and when no angle is greater than 120° it is inside. If we consider a pair of triangles having two sides of one equal to two sides of the other, but the included angles bearing the relation, that in the first it is $120° - \varphi$, and in the second $120° + \varphi$, it can easily be proved that the sum of the distances from the I_+ point to the vertices of the first triangle is equal to the sum of the distances from the I_+ point of the second triangle to the acute angles diminished by the distance to the obtuse angle.

II. CIRCUMSCRIBED AND INSCRIBED TETRAHEDRA.

By ALBERT A. BENNETT, University of Texas.

In this MONTHLY (*1923*, 178), Professor Coolidge mentions as "Problem 8" among "Some unsolved problems in solid geometry," the following: "What relation must exist between spheres in order that it may be possible to inscribe a tetrahedron in one which is circumscribed to the other?"

One might anticipate applications of elliptic or of hyperelliptic functions as interesting as those in the plane case; that is, until one looks at the problem more closely and notes the presence of too many degrees of freedom. The fact is that the problem is far from poristic. Analytic methods are unwieldy unless they follow closely along the lines of possible synthetic methods. The important theorem in this relation is capable of simple expression as follows.

Theorem: Given any sphere, S, and any point, C, within the sphere. For any two distinct points, P and Q, whose join does not pass through C, there is a length, R, such that for every sphere, S′, with center C, and radius less than R, there will be exactly two distinct tetrahedra inscribed in S and circumscribed about S′ and having P and Q for two of the vertices. (The term "circumsribed" is used in its narrow sense, as distinguished from "escribed.")

In the following synthetic proof, the explicit formulation of R in terms of S, C, P, and Q, is not made, but the magnitude of the radius is restricted by the order relations in resulting constructions. Consider a small sphere, S', with C as center, just how small, will appear later. Draw two tangent planes, α and β, both of which pass through P and Q and tangent to S' at T_A and T_B, respectively. Consider the self-inverse projectivity π_P, defined between points of the circle, A, in which α cuts S, and points of the circle, B, in which β cuts S, as follows. Two points, one of A and one of B correspond under π_P (neither being P nor Q) if and only if the plane determined by these two points and P is tangent to the sphere, S'. The special cases of P and Q are easily handled by continuity. There will be of course two planes tangent to S' through P and any given point of A. However α is itself one such plane, and the other which is thereby uniquely determined is the one employed in the projectivity. The projectivity, π_Q, is defined analogously by the use of Q in place of P. A pair of points, one on A and one on B, which correspond under both π_P and π_Q will serve as vertices with P and Q of the tetrahedron desired, provided that the tetrahedron thus obtained, whose faces are tangent to S', actually contains S' in its interior. The product transformation, $\pi_P\pi_Q$, will be for each of the circles, A and B, a projectivity of the figure into itself, and will have two real, one real, or two imaginary fixed points. It is necessary to show that two real solutions are always obtained, actually containing S' within the tetrahedron, provided only that S' be taken with less than a determinate radius.

We shall make use of certain particularly labelled points and it is suggested that the reader make his own illustrations, labelling the proper points as directed. Denote by D_{AP} the other intersection with A of the line PT_A. In a similar manner, define D_{AQ}, D_{BP}, D_{BQ}. Of the two planes through P tangent to S' and perpendicular to the plane PQC, let that one which does not separate Q from C, cut circle A again in E_{AP}. Define in similar manner, E_{BP}, E_{AQ}, E_{BQ}. The points, P, D_{AQ}, D_{AP}, Q, will occur on A in this cyclic order, since the lines used to determine D_{AQ} and D_{AP} intersect at T_A which is within the circle A. The point E_{AP} is between P and D_{AP}, and may be made to come as close to the latter as may be desired if the radius of S' be taken sufficiently small. Our restriction upon this radius will now be expressed by the conditions that the points on A and B occur in the following cyclic orders: on A, we are to have the order, P, D_{AQ}, E_{AQ}, E_{AP}, D_{AP}, Q, and similarly on B. The transformation, π_P, when used from A to B, carries P as a point of A into D_{BP} of B, E_{AP} of A into E_{BP} of B, D_{AP} of A into P as a point of B, Q as a point of A into Q as a point of B. In particular, π_P carries all the points (including D_{AQ}, E_{AQ}, E_{AP}) of the arc PD_{AP}

into points of the arc $D_{BP}P$ (including E_{BP}, E_{BQ}, D_{BP}) of the circle B, but in such a way that the points of the large arc PE_{AP} of A go into the points of the small arc, $D_{BP}E_{BP}$ of B, while points of the small arc, $D_{AP}E_{AP}$ of A go into points of the large arc, PE_{BP} of B. The transformation π_Q, may be discussed in a similar manner as to its effect from B to A.

Consider now what happens when a point of A is carried from A to B by π_P, and this is in turn carried back to A by π_Q. The points of the arc PE_{AP} of A are first carried into those of the small arc, $D_{BP}E_{BP}$ of B. But this arc lies entirely within the arc QE_{BQ} of B, which latter is carried by π_Q into $D_{AQ}E_{AQ}$. Thus the result of the two transformations performed in succession is to carry PE_{AQ} into an arc which lies entirely within it. It follows from the continuity of the projectivity, that there will be some point of arc $D_{AQ}E_{AQ}$ which is carried into itself. Since this point will lie between D_{AQ} and E_{AQ}, and will correspond under both transformations with a point of B between D_{BP} and E_{BP}, this pair of corresponding points furnishes an actual solution of the problem, and does not correspond to an escribed tetrahedron. Likewise the small arc $D_{AP}E_{AP}$ is carried as a result of the two transformations applied successively into the arc QE_{AQ}, which completely envelopes it. It therefore also contains an invariant point which furnishes a second actual solution of the problem, as promised.

III. On Algorithms for the Solution of the Linear Congruence.

By H. S. Vandiver, Cornell University.

The purpose of this note is to call attention to an algorithm for the solution of the linear congruence,

$$ax \equiv 1 \pmod{m}, \tag{1}$$

a and m being positive integers prime to each other, which differs from the well-known method depending on the expansion of m/a as a continued fraction. It appears from an examination of Dickson's *History of the Theory of Numbers*, volume II, chapter 2, that the only papers cited there which deal to any extent with the method I shall describe are due to Binet and Sardi. (Explicit references below.)

1. Assume first that $m = p^\alpha$, where p is prime. Consider

$$k_1 m = aq_1' + r_1'$$

k_1 being any integer $\neq 0$, and $|r_1'| < a$. If r_1' is divisible by p, put

$$k_1 m = a(q_1' \mp 1) + r_1' \pm a,$$

where $|r_1' \pm a| < a$. Now $r_1' \pm a$ is prime to p, for if we assume $r_1' \pm a \equiv 0 \pmod{p}$, then $a \equiv 0 \pmod{p}$, contrary to hypothesis. Hence we may write in any case

$$k_1 m = aq_1 + r_1,$$

where $|r_1| < a_1$ and r_1 is prime to m.

Similarly we have

$$k_2 m = r_1 q_2 + r_2,$$

where $|r_2| < |r_1|$ and r_2 is prime to m, k_2 being any integer $\neq 0$. Proceeding in this way we obtain ultimately

$$k_s m = r_{s-1} q_s + r_s,$$

where $r_s = \pm 1$. Hence in general we have

$$k_i m = r_{i-1} q_i + r_i,$$

$i = 1, 2, \cdots, s$, and $r_0 = a$, $r_s = \pm 1$. Also k_i is any integer, not zero, and r_i is prime to m.

We may write then

$$r_{i-1} q_i \equiv -r_i \pmod{m}.$$

Setting $i = 1, 2, \cdots, s$, and multiplying all the congruences together we have

$$a \prod_i q_i \prod_{c=1}^{s-1} r_c \equiv (-1)^s r_s \prod_{c=1}^{s-1} r_c \pmod{m}.$$

Dividing through by $\prod_{c=1}^{s-1} r_c$ which is prime to m, we have

$$a \prod_i q_i \equiv (-1)^s r_s \pmod{m}.$$

So that

$$x = (-1)^s \frac{\prod_i q_i}{r_s}$$

2. For the case $m = p$, a prime, the algorithm takes a simple form since any remainder r after p is divided by any integer less than it is necessarily prime to p. If in this case we put all the k's equal to unity and select all the r's positive then we obtain the method given by Binet,[1] and Sardi.[2]

3. For m divisible by two or more distinct primes it is generally possible in practice to select quickly an integer k so that

$$km = aq + r$$

where $|r| < a$ and r is prime to m, in which case we proceed as before. But when m is divisible by more than two *small* primes, the selection of k may become difficult and we may proceed as follows:

Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_e^{\alpha_e}$, where the p's are primes, and assume that $p_i^{\beta_i}$ is the highest power of p_i which is less than a, $i = 1, 2, \cdots, e$.

Let $M = \prod_{i=1}^{e} p_i^{\beta_i}$. Then

$$Mm = aq + r$$

where $|r| < a$. If r is not prime to m we may write $r = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_e^{\gamma_e} r_1$,

[1] *Comptes Rendus*, Paris, 1841, 210–3. Binet is interested in finding both x and y in $ax = 1 + my$, so that his work is unnecessarily complicated from the standpoint of the theory of congruences, as the congruence notation is not used.

[2] *Giornale di Matematiche*, 7, 1869, 115–6.

where r_1 is prime to m and $\gamma_i \leqq \beta_i$. Hence r/r_1 divides M and we have

$$M_1 m = aq_1 + r_1.$$

In a similar way we obtain

$$M_2 m = r_1 q_2 + r_2$$

where $|r_2| < |r_1|$, and r_2 is prime to m, and proceeding as before we obtain a solution of (1).

The latter method, although direct, involves long, impracticable computations at times.

For m composite we may also use the method of Binet,[1] which may be extended and made more elastic as indicated in the above discussion.

Example 1. Solve $81x \equiv 1 \pmod{257}$.

We have

$$\begin{aligned} 257 &= 81 \times 3 + 14, \\ 257 &= 14 \times 18 + 5, \\ 257 &= 5 \times 51 + 2, \\ 257 &= 2 \times 128 + 1. \end{aligned}$$

Hence

$$81(3 \cdot 18 \cdot 51 \cdot 128) \equiv 1 \pmod{257}.$$

In setting up the algorithm, we noted that the k's were arbitrary, so we may select them so as to make the remainders as small as possible. For example, from

$$257 = 81 \times 3 + 14$$

we see that 81 is close to 6×14, so we use

$$\begin{aligned} 6 \times 257 &= 81 \times 19 + 3, \\ 257 &= 3 \times 86 - 1. \end{aligned}$$

Whence

$$\begin{aligned} 81(19 \cdot 86) &\equiv -1 \pmod{257}, \\ 81 \cdot 165 &\equiv 1 \pmod{257}. \end{aligned}$$

Example 2. Solve $31x \equiv 1 \pmod{120}$.

$$\begin{aligned} 2 \cdot 120 &= 7 \cdot 31 + 23, \\ 2 \cdot 120 &= 23 \cdot 11 - 13, \\ 2 \cdot 120 &= 13 \cdot 19 - 7, \\ 120 &= 7 \cdot 17 + 1. \end{aligned}$$

Hence

$$31(7 \cdot 11 \cdot 19 \cdot 17) \equiv 1 \pmod{120}.$$

[1] *Journal de Math. (Liouville)*, 1841, I, 6, 491–6.

Example 3. Solve $1723x \equiv 1 \pmod{4028}$.

Without setting down the actual figures, we see immediately that the remainder when 4028 is divided by 1723 is close to 1/3 of 1723. Hence we set

$$\begin{aligned} 3 \times 4028 &= 1723 \times 7 + 23, \\ 4028 &= 23 \times 175 + 3, \\ 4028 &= 3 \times 1343 - 1; \end{aligned}$$

and we have

$$1723(7 \cdot 175 \cdot 1343) \equiv 1 \pmod{4028}.$$

In order to compare the amount of computation in this process with that involved in the use of the continued fraction algorithm, we set down the relations required in the latter:

$$\begin{aligned} 4028 &= 1723 \times 2 + 582, \\ 1723 &= 582 \times 3 - 23, \\ 582 &= 23 \times 25 + 7, \\ 23 &= 7 \times 3 + 2, \\ 7 &= 2 \times 3 + 1. \end{aligned}$$

x is the numerator of

$$2 + \cfrac{1}{3 - \cfrac{1}{25 + \cfrac{1}{3 + \cfrac{1}{3}}}}.$$

In most cases that I have examined, the comparison is about the same as in this example.

RECENT PUBLICATIONS.

EDITED BY D. C. GILLESPIE, Cornell University, to whom communications should be sent.

REVIEWS.

From Determinant to Tensor. By W. F. SHEPPARD. Oxford University Press (American Branch, New York). 1923. 12mo. 127 pages. Price $2.85.

This book might be characterized as a remarkably simple and lucid introduction to the subject of tensors with particular emphasis upon the notation as used by Eddington. The operations upon tensors which are considered are the obvious algebraic ones including that of obtaining the inner product, and the operation of differentiation. No mention is made of the Christoffel symbols and the physical significance of tensors is only suggested. On the completion of the reading of this book, the student will have reached a stage toward which Eddington, for example in his *Mathematical Theory of Relativity*, devotes approximately ten pages. This book is not verbose, does not discuss proofs nor enter into philosophical speculations. The difference in the number of pages may be accounted for in part by the choice for this volume of a small format, and in part by the chapter devoted to an application of tensors to statistics, but chiefly by the detailed character of the explanations and warnings involved in developing

the subject from elementary notions. This may be most easily seen perhaps from the fact that about half of the book is spent in developing the essentials of the theory of determinants and matrices from a point of view slightly more abstract than usual.

As an elementary exposition, this text is highly successful. It contains no exercises and makes no claim to be an embodiment of advanced research or a reference book for theorems or formulas. A fair notion of the content, purpose and general character of the book can be obtained from the author's preface most of which will now be quoted.

"The tensor calculus used in the mathematical treatment of relativity, and concisely explained by Professor A. S. Eddington in his '*Report on the Relativity Theory of Gravitation*,' is, like the various kinds of vector calculus, a system of condensed notation which not only conduces to economy in the writing of symbols, but, what is more important, enables spatial and physical relationships to be grasped as a whole without having to be built up from a number of components which really represent views of different parts of space. Three-dimensional geometry or physics is troublesome enough; the addition of a fourth dimension made the need of a condensed notation imperative.

"Professor Eddington has recently pointed out that the tensor notation and methods can be applied, with happy results, to other and more elementary classes of problems than those for which they were originally devised; and this book is an attempt to put his somewhat compressed exposition into a form in which it will appeal to a larger circle of readers. The book, therefore, is not intended as an introduction to the mathematical theory of relativity—though I hope it may be of some use for that purpose—but rather as an exercise in the elementary application in the methods which, apart from any practical use, possess a special beauty of their own.

"The new notation is not introduced until the fifth chapter. The properties of determinants, which serve as the starting point for the application of the notation, are familiar to the mathematician; but, as I hope the book may be read by some who are not entirely at ease with determinants, I have commenced with four chapters on the elementary theory of the subject. . . .

"What I have called double sets will be recognized by the advanced student as matrices, and many of the propositions will be found to be familiar. But the tensor calculus may fairly claim that, in bringing into closer relation various branches of mathematical study, previously regarded as distinct, it gives them a new life."

ALBERT A. BENNETT.

Géométrie Générale Synthétique Moderne. By ÉMILE BALLY. Paris, Gauthier-Villars et Cie. 1922. Paper, 8vo. viii + 218 pages.

Here is a book from the island of Martinique "dedicated to the friends of geometry . . . written by an amateur for amateurs . . . with a desire to please as well as to convince," available, the author hopes, for readers of intelligence, whether mathematicians or not, who have some taste for abstract reasoning. The author has had to omit extensive references to the work of others, his library being made up of the work of Darboux (*Classe remarquable de courbes et de surfaces*), of Dumant (*Surfaces cubiques*), of Duporcq (*Géométrie moderne*) and the *Encyclopédie des Sciences mathématiques*. As if the lack of tools were not sufficient handicap, the author has had to adjust his work to the higher cost of printing, so that the present volume contains only the preliminary chapter on ordinal arithmetic, chapter I, on the foundations of general geometry, chapters XII, XIII, XIV, on the study of the hexagon and allied configurations, together

with a short appendix containing certain corrections and additions to his work on the synthetic geometry of unicursal curves of the third class and fourth order published in 1920.

The first chapter on ordinal arithmetic contains an exposition of the theory of the number system and of the continuum of points in a line. Just how necessary this chapter may have been for the rest of the work is difficult to judge without examination of the ten missing chapters. The subject is so well treated elsewhere, however, that in the opinion of the reviewer it might have been replaced to advantage by some of the chapters on geometry which have been crowded out.

Chapter I undertakes to give a synthetic development of n-dimensional geometry. It is difficult to read, even for one who is fairly familiar with the subject, on account of the long list of unfamiliar terms. This coining of new terms seems to be characteristic of workers in synthetic geometry. Of the seventy or more new expressions invented by Desargues only the one: "involution" seems to have survived. A glance at the headings of each division of this chapter conveys little to the ordinary mathematician. One must get familiar with the meaning of such terms as "polynarite," "soutiens," "formes axées," "anaxes," "feuillées," "iso-similaire," "punctidualilinéaire," "dualisimilaire." The chief result of the first division is: "Between the polynarities m, and n, of two elements and the polynarities i and j of their intersection and junction exists always the fundamental relation $m + n = i + j$." Translated into ordinary language, this is the familiar theorem about linear spaces S_r: "If S_{r_1} and S_{r_2} intersect in an S_a they lie in an $S_{r_1+r_2-a}$.

Again one is at a loss to judge the necessity of chapter I for the remainder of the book without some knowledge of the contents of the missing chapters. Chapters XII, XIII, and XIV which have to do with configurations in the plane do not need so formidable an introduction. Chapter XII furnishes a detailed study of the various points and lines connected with a hexagon. An elaborate notation is developed to indicate the sets of lines and points. The various kinds of grouping of points and lines are given names and enumerated. All this without supposing the six points to be related in any way. In the final chapter, the hexagon is made to satisfy certain conditions equivalent to the supposition that its vertices lie on a conic. The usual theorems of alignment and concurrence which hold for Pascal lines, Steiner points, Kirkman points; Salmon-Cayley lines, etc., are given (in unfamiliar language) together with theorems affecting certain other elements, not so well known, which the author names for himself, "the points and lines of Bally."

The author has undertaken a useful piece of work in bringing together the scattered results connected with the configuration of six points on a conic and if the reader were not confronted at each step by "textures" and "skeletons" and other strange things it would be all that M. Bally has hoped for it.

D. N. LEHMER.

Les Applications Élémentaires des Fonctions Hyperboliques à la Science de l'Ingénieur Électricien. By A. E. KENNELLY. Paris, Gauthier-Villars, 1922. 153 pp.

This book covers the scope and purport of a series of public lectures delivered in France by the author while serving as the first exchange professor representing a group of American Universities in certain French Universities. The subject matter is practically the same as a previous work in English by the same author —viz: "*The Application of Hyperbolic Functions to Electrical Engineering Problems*" published in 1912 by the University of London Press. It is unfortunate that no reference is made in the preface to this earlier publication. This earlier work is fuller and more complete and would be preferred by American students.

Professor Kennelly was among the first to advocate strongly the wider use of the complex quantity in electrical engineering problems. In considering problems involving long electrical lines where it is necessary to take into account the uniform linear distribution of the resistance, leakage conductance, inductance and capacitance the rigorous solution leads to hyperbolic functions.

The author bases his treatment upon the fact that the theories of continuous currents and alternating currents are essentially the same; all continuous current formulas holding for alternating current circuits when complex numbers are substituted for real numbers. The practical application of hyperbolic functions to engineering problems was handicapped by the lack of suitable tables until this need was supplied by Professor Kennelly with "*Tables of Complex Hyperbolic and Circular Functions*" published in 1914 by the Harvard University Press. Reference to this work should also have been made in the preface.

In developing the idea of hyperbolic functions the author defines the magnitude of a hyperbolic angle, first by the ratio of the hyperbolic arc distance described to the length of the radius vector, and second by the area of the hyperbolic sector; and then traces the relationship between the circular and hyperbolic functions. The functions of a complex angle are derived from a mixed circle and hyperbola diagram. For the mathematician, the main interest in the book would be as a source for problems showing the application of elementary hyperbolic functions to certain problems in electrical engineering.

JOSEPH H. CANNON (University of Michigan).

Mathematical Theory of Finance. By T. M. PUTNAM. New York, John Wiley and Sons, 1923. 8vo. 10 + 117 pages. Price $1.75, postpaid.

It is stated in the preface: "The scope and method of the book have been designed for a three-hour course for one semester, such as is prescribed in the College of Commerce in the University of California."

For a short course the book is an excellent text. The author has used good judgment both in including important formulas and in excluding those of minor importance. In particular, the cumbrous and unnecessary formulas for the time n are conspicuously absent. But when, as on page 18, a problem is given to find n, the problem is easily solved; and moreover, a proper interpretation is given to the fraction which appears.

The real significance of formulas is pointed out by frequent verbal interpretations; and the interpretations given are very good. Perhaps a more simple interpretation of the formula for capitalized cost than that given on page 29 is obtained by noting that the interest on a unit of money for k years is $(1 + i)^k - 1$; and thus the reciprocal of this expression will yield one unit of money every k years.

Formulas for nominal discount are not developed; but on page 12 a problem involving nominal discount is solved by use of nominal interest. Nominal discount is perhaps more important and more easily understood than the force of interest, treated on pages 8 and 9. But—granting that the force of interest should appear even in a brief course—it is to be noted that the author defines the force of interest clearly as the limiting value of j, "for a given effective rate, i" ... "as m increases without bound." Much confusion arises unless it is recognized that the i is given, *i.e.*, fixed.

The chapter headings are: interest, annuities, amortization—sinking funds, bonds, probability, life annuities, elementary principles of life insurance.

In a brief chapter on probability, designed for practical purposes, it is unnecessary to draw a sharp line between abstract and empirical probability; and this is not done. Unfortunately, the formal definition for probability on page 62 neglects mention of equally likely cases. However, in the explanation that precedes and follows, this restriction is made clear.

Eleven tables are given, including tables for the usual monetary functions, to seven places of decimals, with sixteen different rates of interest, also the American Experience Mortality Table, and Commutation Columns based thereon at $3\frac{1}{2}\%$ interest for D_x, M_x, N_x, the N_x summing from age x on.

The book has been well printed, and is pleasing to the eye. Very few misprints were noticed—the c in the last line on page 16, and the $p + p$ in line 11, page 69, would trouble no one.

In conclusion, the text throughout is characterized by an excellent choice of material and by lucid explanations.

E. L. Dodd.

Chance and Error. By Marsh Hopkins. London, Kegan Paul, Trench & Co., 1923. 223 pages. Price $3.00.

As is stated in the preface "This little book shows that the vagaries of chance are the result of the interference of yes and no. . . . It was written with the object of extending the usefulness of this very important subject to those whose knowledge of mathematics is limited."

The book is made up of five hundred and fifty examples and their solutions. Some of the chapter headings are: games whose expectation is zero, direct observations, indirect observations, statistics, target practice, errors in three dimensions, monte carlo, variable chances.

Exposition of any well-directed theory is lacking. Each example is stated, briefly explained, and solved. Most explanations are satisfactorily clear to one who has a previous knowledge of least squares and statistics; for one not so schooled, I doubt if the book will be satisfactory, certainly not inspiring.

New names are given to old concepts. For example, we find on page 44, "The *median error* is the error that is as likely to be exceeded as not." I see no necessity for a new name for this concept, the most approved expression for which is the *probable error*. A term which the author uses frequently but never carefully defines is the *average error*.

The basis for the author's logic is found at the beginning of chapter one. Some of the definitions are very obscure. For example, "An element of any thing is very small when its magnitude may be disregarded." And "Anything that tends to occur and cannot occur is imaginary."

The book should prove of some value to the "home" student in Probability. Since Whitworth's "*Choice and Chance*" is now out of print, this book may in a measure fill the void.

C. H. RICHARDSON.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Isis, volume 5, no. 13, October, 1922: "Michigan mathematical papyrus no. 621 (with facsimile)" by L. C. Karpinski, 20–25; "Notes on the knowledge of latitudes and longitudes in the Middle Ages" by J. K. Wright, 75–98; "Quadripartitum Ricardi Walynforde de sinibus demonstratis" by J. D. Bond, 99–115; "Mathematical signs of equality" by F. Cajori, 116–125. Volume 5, no. 14, May, 1923: "Richard Wallingford's Quadripartitum" (English translation) by J. D. Bond, 339–363; "Beiträge zur arabischen Trigonometrie" by C. Schoy, 364–399; "Ein wichtiger Satz über die Ellipse des Fagnano und seine Ergänzung" by K. Bopp, 400–402; "Entwickelungslinien in der Geometrie" by K. Bopp, 406–408.

Journal of the Washington Academy of Science, volume 12, no. 19, November 19, 1922: "Values of sine and cosine θ to 33 places of decimals for various values of θ expressed in sexagesimal seconds" by C. E. Van Orstrand and M. A. Shoultes, 424–436. Volume 13, no. 8, April 19, 1923: "A remarkable formula for prime numbers" by Paul R. Heyl, 150–151.

Science Progress, volume 48, no. 69, July, 1923: "Recent advances in science: Mathematics" by F. P. White, 1–5; "Indeterminate equations of the third degree" by L. J. Mordell, 39–55.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I.

[*1923*, 39.]

The printed program of the Mathematics Club of Brown University for 1923–1924 makes the following announcements:

November 2, 1923: "William Rowan Hamilton" by Dempster Hobron '24; "Some properties of the cycloid" by Charlotte Perry '25; "Magic squares" by Frederick Kilbourne, Jr. '26.

December 14: "The map coloring problem" by Mildred Carlen '24; "The history of trigonometry" by Grace Hanson '25.

January 18, 1924: "The five Platonic bodies and higher polyhedra derived from them" by Albert Wheeler, North High School, Worcester, Mass.

February 29: "Blaise Pascal" by Avis Sugden '26; "Cover the red spot" by Frances Wright, Gr.; "Mathematico-chess recreations" by Clarence Bennett, Gr.
March 28: "Repeating decimals" by Professor Mary Curtis Graustein of Wellesley College.
May 2: "The history of mathematics at Brown University" by Elizabeth Stafford, Gr.; "The integraph" by George Sauté '24; "Curiosities in numbers" by Frederick Wood '26.
May Picnic.

The Mathematics Club of Cooper Union, New York City.

[*1923*, 40.]

The following officers served for the year 1922–1923: President, Peter Kosting '25; vice-president, Barnett Emmerich '23; secretary, Fred Miller '26; faculty advisers, Professor H. W. Reddick and H. W. Barcus, instructor. A membership fee of twenty-five cents for the year provided a fund for a prize to be awarded at commencement to the member of the first-year class having the highest average in mathematics. This prize, a polyphase duplex slide-rule, was won by Fred Miller '26.

Meetings were held on alternate Mondays as follows:
October 23, 1922: "Explanation of the slide-rule" by H. H. Barcus.
November 6: "History of our number system" by Peter Kosting '25.
November 20: "Mathematical fallacies" by Fred Buhrendorff '25.
December 4: "Theorems on collinear points" by Fred Miller '26.
December 18: "Flatland" by David Samson '24.
January 8, 1923: "Hyperspace" by Professor Reddick.
January 22: "Inscribing a cylinder of maximum volume in a cone" by Alexander Gotsdanker '24.
February 5: "Construction of regular polygons" by Fred Van der Voort '26.
February 19: "Demonstration of calculating machines" by Mr. Coxhead, of the Mercedes Calculating Machine Co.
March 5: "Sailing faster than the wind" by Fred Buhrendorff '25; "Solving the right triangle without tables" by Barnett Emmerich '25.
March 19: "Mathematical anecdotes" by Jacob Boorstein '25.
April 2: "An original approximate method of trisecting an angle" by William J. Pickett, instructor.
April 16: "Geometric proofs of the law of tangents" by Isidore Fankucken.

(Report by Mr. Miller.)

The Mathematics Club of Hunter College, New York City.

[*1922*, 354.]

The officers of the Mathematics Club of Hunter College for the year 1922–1923 were: President, Sara Malkin '23; vice-president, Isabel Graves '23; secretary, Esther Alfert '24; treasurer, Bessie Schoenfeld '23; faculty adviser, Miss H. Kunte.

The following topics were presented at the meetings:
The three problems of antiquity—"Duplication of the cube—Cissoid of Diocles" by Lillian Lesser '24; "Trisection of the angle—Conchoid" by Bessie Schoenfeld '23; "Squaring the circle—Quadratrix" by Miriam Jacobi '23.
"Geometry of the compass" by Sylvia Rosenstein '26.
"The construction of magic squares" by Isabel Graves '23.
"Japanese and Chinese mathematics" by Sara Malkin '23.
"How we reckon time" by Harriet Griffin '25.
"Linkages and their applications" by Eugenie Schein '24.
"Mathematics from Ahmes to the Renaissance" by Professor Emma M. Requa.

(Report by Miss Alfert.)

The Mathematics Club of the University of Nebraska, Lincoln, Neb.

[*1920*, 320.]

The papers read at meetings in 1922–1923 were as follows:
November 8, 1922: "Some card tricks and the explanation" by Dean C. Engberg.
December 13: "History of the development of mathematics" by Dean A. L. Candy.

January 10, 1923: "Life of Archimedes" by Dean Candy; "What do the letters of algebra stand for?" by Daisy Portenier; "Short cuts in arithmetic" by R. G. Sturm.
February 14: "The Youth Movement in Holland" by Pit Roest.
March 14: "Paper folding" by Gerold Almy; "Archimedes—his method" by Dean Candy.
April 11, 1923: "Einstein's theory" by E. Z. Stowell; "Development of logarithms" by Dean Candy.
May 9: "The story of the ten digits (illustrated by lantern slides)" by Dean Candy.

(Report by Vivian Hanson, Secretary.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3062. Proposed by HARRY LANGMAN, New York City.

Let A be the vertex of a hyperbola. Draw APQ, any line through A cutting the curve again in P and either asymptote in Q. Draw PR parallel to the other asymptote, cutting the first in R. Show that the length RQ is constant and give a simple mechanical construction of the curve based on this property.

3063. Proposed by H. A. BENDER, University of Illinois.

Show that

$$\sum_{r=1}^{r=b}(p^a-1)(p^a-p)\cdots(p^a-p^{r-1})\frac{(p^b-1)(p^{b-1}-1)\cdots(p^{b-r+1}-1)}{(p^r-1)(p^{r-1}-1)\cdots p-1}=p^{ab}-1, \quad (a \geqq b)$$

is an identity in p.

3064. Proposed by BURRELL MORGAN, Panther, W. Va.

One of the parallel sides of a trapezoid containing $17\frac{1}{2}$ acres is 80 rods and the two non-parallel sides are 22 and 48 rods respectively; what is the length of the remaining side?

3065. Proposed by A. S. WIENER, Cornell University.

Prove the following identity:

$$\begin{vmatrix} a^{m+1}+c^m(a+b), & b(bc^{m-1}+a^m), & b(a^{m-1}c+a^m+c^m) \\ c(ab^{m-1}+a^m+b^m), & b^{m+1}+a^m(b+c), & c(a^{m-1}c+b^m) \\ a(ab^{m-1}+c^m), & a(bc^{m-1}+b^m+c^m), & c^{m+1}+b^m(c+a) \end{vmatrix}$$

$$\times \begin{vmatrix} a^{n+1}+c^n(a+b), & b(bc^{n-1}+a^n), & b(a^{n-1}c+a^n+c^n) \\ c(ab^{n-1}+a^n+b^n), & b^{n+1}+a^n(b+c), & c(a^{n-1}c+b^n) \\ a(ab^{n-1}+c^n), & a(bc^{n-1}+b^n+c^n), & c^{n+1}+b^n(c+a) \end{vmatrix}$$

$$= 8\begin{vmatrix} a^{m+n+2}+c^{m+n}(a^2+b^2), & b^2(b^2c^{m+n-2}+a^{m+n}), & b^2(a^{m+n-2}c^2+a^{m+n}+c^{m+n}) \\ c^2(a^2b^{m+n-2}+a^{m+n}+b^{m+n}), & b^{m+n+2}+a^{m+n}(b^2+c^2), & c^2(a^{m+n-2}c^2+b^{m+n}) \\ a^2(a^2b^{m+n-2}+c^{m+n}), & a^2(b^2c^{m+n-2}+b^{m+n}+c^{m+n}), & c^{m+n+2}+b^{m+n}(c^2+a^2) \end{vmatrix}.$$

3066. Proposed by B. F. FINKEL, Drury College.

What is the amount of work done in pulling a spool of thread, weight w, up an inclined plane whose length is l and inclination α, the spool to be pulled up the plane by taking hold of the outer end of the thread and allowing the thread to unwind? We assume that we may neglect the weight of the thread unwound, that friction is large enough to prevent slipping, and that the axis of the spool remains horizontal.

3067. Proposed by J. H. MURPHY, Pittsburgh, Pa.

On the base of a right triangle, whose altitude, a, is greater than the base, b, is constructed a triangle whose vertex angle is α. What are the lengths of the two variable sides of this triangle when that part of its area outside of the right triangle is a maximum?

3068. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that

$$\sum_{n=1}^{n=\infty} u_n/n! = (e-1)^2/2$$

if $u_1 = 0$, $u_n = 2u_{n-1} + 1$, $(n = 2, 3, 4, \cdots)$.

3069. Proposed by J. ROSENBAUM, Milford, Conn.

Given the mid-points of the sides of a quadrilateral inscribed in a circle and the radius of the circle, to construct the quadrilateral.

SOLUTIONS.

2998 [1922, 420]. Proposed by F. M. GARNETT, Augusta, Georgia.

A cube has removed from it a right pyramid whose base is a face of the cube and whose altitude is the altitude of the cube. How far from the base of the cube must a plane be passed parallel to the removed face so as to divide the remaining volume of the cube into two equal parts?

SOLUTION BY THEODORE BENNETT, University of Illinois.

Let an edge of the cube be a. The volume of the pyramid is $\frac{1}{3}a^3$, and the remaining volume is $\frac{2}{3}a^3$. Cut the remaining solid by a plane parallel to the base and a distance x above it. The area of the section is clearly

$$a^2 - (a-x)^2 = 2ax - x^2.$$

Hence the volume which we have cut off is

$$\int_0^x (2ax - x^2)dx = ax^2 - \frac{x^3}{3}.$$

We wish to determine x so that

$$ax^2 - \frac{x^3}{3} = \frac{a^3}{3}.$$

The solutions of this equation are

$$a(1 - 2\sin 10°), \qquad a(1 - 2\sin 50°), \qquad a(1 + 2\sin 70°).$$

The first of these represents the solution of our problem, being approximately $.65a$.

NOTE BY THE EDITORS:—The same result is obtained if the cube is replaced by any cylinder or prism of altitude a and area of base A. It is simpler to compute the volume of the other portion of the figure. If y is the distance of the plane from the vertex, the volume of this portion is the difference between that of a cylinder of base A and of a cone of base $A(y/a)^2$, hence

$$Ay - \frac{Ay^3}{3a^2} = \frac{Aa}{3} \qquad \text{or} \qquad y^3 - 3a^2y + a^3 = 0.$$

The roots of this equation are $2a\sin 10°$, $2a\sin 50°$, $-2a\sin 70°$, as we see at once by comparing it with the identity $4\sin^3\theta - 3\sin\theta + \sin 3\theta = 0$. The first root is the solution of the problem.

Also solved by H. N. CARLETON, W. F. DANTZSCHER, PHILIP FITCH, MICHAEL GOLDBERG, H. HALPERIN, H. A. ROBINSON, and W. W. WEBER.

2999 [1923, 41]. Proposed by M. B. PORTER, University of Texas.

Given n positive numbers: $a_1, a_2, a_3, \cdots, a_n$, then

$$\Sigma a_i \Sigma a_j^{-1} > n^2 \qquad (i, j = 1, 2, 3 \cdots n)$$

unless $a_i = a_j$, for which case equality occurs; show by passing to limits that, if $\varphi(x) > 0$ and is continuous, $\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)}$ takes on its minimum for $\varphi(x) =$ constant over the interval a to b.

Partial Solution by W. M. Whyburn, University of Texas.

The inequality may be proved by observing that

$$\Sigma a_i \Sigma a_j^{-1} = n + \Sigma \left(\frac{a_i}{a_j} + \frac{a_j}{a_i} \right), \tag{1}$$

where in the summation on the right $i \neq j$ and there are $n(n-1)/2$ terms. If $a_i \neq a_j$, then $(a_i - a_j)^2 = a_i^2 + a_j^2 - 2a_i a_j > 0$; and, since the a's are greater than zero, we have $a_i/a_j + a_j/a_i > 2$. Hence, if not all the a's are equal, we deduce from (1)

$$\Sigma a_i \Sigma a_j^{-1} > n^2, \tag{2}$$

but if all the a's are equal, the two sides of the above are equal.

If the interval from a to b is divided into n equal sub-intervals by $x_1, x_2, x_3, \cdots, x_n = b$; and, if we set $a_i = \varphi(x_i)$, then

$$\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)} = \lim_{n=\infty} \frac{(b-a)^2}{n^2} \Sigma a_i \Sigma a_j^{-1}, \tag{3}$$

since $\varphi(x)$ is continuous and does not vanish within or at the ends of the interval considered. Hence by (2)

$$\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)} \geqq (b-a)^2. \tag{4}$$

If $\varphi(x)$ is a constant, *i.e.*, all the a's are equal, the equality sign holds. Thus the minimum of the left side of (4) is $(b-a)^2$ and this minimum is reached when $\varphi(x)$ is any constant not zero.

Note on the above Solution by Otto Dunkel, Washington University.

The proof above shows that the minimum of the left side of (4) is reached when φ is a constant, but it does not show that this is true *only* in that case. For we cannot conclude by the above reasoning that, if φ is not a constant, the inequality above holds true in (4). To conclude this we should show that, when the a's are not all equal, the left side of (2) is greater than pn^2, where p is a fixed number greater than unity. The proof may be completed as follows: The right side of (1) may be written

$$n^2 + \Sigma \frac{(a_i - a_j)^2}{a_i a_j}.$$

Hence

$$\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)} = (b-a)^2 + \int_a^b \int_a^b \frac{[\varphi(x) - \varphi(y)]^2}{\varphi(x)\varphi(y)} dxdy. \tag{6}$$

If $\varphi(x)$ is not a constant in the interval considered, then for some pair of values, α and β $(a < \alpha < b, a < \beta < b)$ $\varphi(\alpha) \neq \varphi(\beta)$. Since $\varphi(x)$ is continuous we can determine in the region $a \leqq x \leqq b$, $a \leqq y \leqq b$, a small region containing (α, β) throughout which the integrand on the right in (6) is greater than some fixed number $m > 0$. If the area of this region is A, then the left side of (6) is greater than $(b-a)^2 + Am$. It then follows that the left side of (6) is greater than $(b-a)^2$ if $\varphi(x)$ is not a constant and equal to $(b-a)^2$ if $\varphi(x)$ is a constant.

The important rôle of the double integral in (6) in the above proof suggests the following simplification:

Consider the integral

$$\begin{aligned} \int_a^b \int_a^b \frac{[\varphi(x) - \varphi(y)]^2}{\varphi(x)\varphi(y)} dxdy &= \int_a^b \int_a^b \frac{\varphi(x)}{\varphi(y)} dxdy + \int_a^b \frac{\varphi(y)}{\varphi(x)} dxdy - 2(b-a)^2, \\ &= 2\int_a^b \int_a^b \frac{\varphi(x)}{\varphi(y)} dxdy - 2(b-a)^2. \end{aligned} \tag{7}$$

We suppose that the same conditions as before are imposed upon $\varphi(x)$. The second form of the right side in (7) follows from an interchange of the letters x and y. By the process of evaluating

a double integral, the last result in (7) may be written

$$2\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)} - 2(b-a)^2. \tag{8}$$

By the proof previously given the left side of (7) is greater than a fixed number which is not zero, if $\varphi(x)$ is not constant: it is obviously zero if $\varphi(x)$ is a constant. Hence (8) is subject to the same conditions, and we have

$$\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)} \geqq (b-a)^2,$$

where the equality sign holds *only* when $\varphi(x)$ is a constant.

Also solved by D. F. BARROW, H. HALPERIN, and A. PELLETIER.

3001 [1923, 41]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

In the plane of a given circle a second circle with a given radius is drawn so that the radical axis of the two circles passes through a given fixed point. Find the locus of the center of the second circle.

SOLUTION BY MABEL M. YOUNG, Wellesley College.

Take the center of the given circle with radius c as origin; (x', y') the coördinates of the fixed point; and (h, k) the coördinates of the center of the second circle of radius r. Then the condition that the radical axis of the two circles $x^2 + y^2 = c^2$ and $(x-h)^2 + (y-k)^2 = r^2$ shall pass through the fixed point is $2x'h - h^2 + 2ky' - k^2 = c^2 - r^2$. But this is the equation of a circle with center (x', y') and radius $\sqrt{r^2 - c^2 + x'^2 + y'^2}$.

This circle, which is the required locus, may always be constructed for all values of r, c, (x', y'), save that when $c > r$, the fixed point may not lie nearer the center of the given circle than $\sqrt{c^2 - r^2}$.

Also solved by S. E. FIELD, L. O. GHORMLEY, MICHAEL GOLDBERG, A. M. HARDING, WILLIAM HOOVER, H. HALPERIN, R. M. MATHEWS, A. PELLETIER, J. B. REYNOLDS, and A. V. RICHARDSON.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

At the fall meeting of the Mathematics Section of the Virginia State Teachers Association, in Richmond, on November 28th, Professor T. McN. SIMPSON, Jr., of Randolph-Macon College, read a paper on the "Social significance of mathematics"; Professor GILLIE A. LAREW, of Randolph-Macon Woman's College, discussed minimum requirements in college mathematics from the standpoint of the actual practice in Virginia and from that of an ideal curriculum; and Dr. H. A. CONVERSE, of the Harrisonburg State Normal School, spoke on the relation of mathematics to other sciences. For the year 1923–1924, Dr. Converse is president and Professor Larew secretary of the Section.

Professor A. E. WHITE, of the Kansas State Agricultural College, spoke before the mathematical round table of the Kansas State Teachers Association at Wichita on October 17th on "Examinations in high school mathematics." On the following day, Mr. W. C. JANES read a paper on "Some geometrical tests" before the corresponding body at Topeka.

At a recent meeting of the Nebraska Academy of Science, Professor R. M. McDILL of Hastings College read a paper before the Mathematics Section on the topic, "Are disciplinary values a legitimate aim to be stressed in the teaching of mathematics?"

Dean J. N. HART, of the University of Maine, who has been head of the department of mathematics for the past thirty years, has been granted a year's leave of absence by the trustees.

Professor W. F. OSGOOD, of Harvard University, has been elected a member of the Leopoldinisch-Carolinisch Deutsche Akademie der Naturforscher.

Miss ETHEL B. CALLAHAN, of the University of Wisconsin, has been appointed head of the department of mathematics at Cedar Crest College, Allentown, Pa.

Mr. W. J. WAGNER has been appointed instructor of mathematics at Allegheny College.

Mr. C. C. WAGNER, of Allegheny College, has been appointed assistant professor of mathematics at Pennsylvania State College, and is now acting head of the department.

Mrs. ETHEL S. KERSHNER has been appointed assistant professor of mathematics at Lynchburg College.

At the University of North Carolina, Mr. S. B. SMITHEY and Mr. L. M. SAHAG have been appointed instructors of mathematics.

At Elon College (North Carolina), Miss LOUISE SAVAGE has been appointed instructor of mathematics. Professor T. C. AMICK has been elected president of the department of higher education of the North Carolina Education Association for the coming year.

At Queen's College, Charlotte, North Carolina, Miss EDNA BERKELE has resigned as head of the department of mathematics, and Miss OLIVE M. JONES has been appointed to that position.

At Chicora College, Columbia, South Carolina, Mr. W. H. MILNER, of the George Peabody College for Teachers, has succeeded Miss JULIA B. PROSSER as head of the department of mathematics.

Miss ANNIE R. ALFORD, of the University of Oklahoma, has been appointed instructor of mathematics at Coker College, Hartsville, South Carolina.

At the Florida State College for Women, Assistant Professor OLGA LARSON has been granted leave of absence, and is at the University of Missouri. Miss Larsen's place is being filled by Miss MYRA B. KEARNEY.

Mr. J. A. HYDEN, after a year's leave of absence, has been made assistant in mathematics and professor of physics at Maryville College (Tenn.).

Mr. GARRETT VAN DER BORGH is now a member of the mathematical staff at Hope College, Mich.

Miss MARY REICHELDERFER, of the University of Chicago, has been instructor of mathematics at St. Xavier College, Chicago, since September 1922.

At Campion College, Prairie du Chien, Wisconsin, Professor J. H. MAY has succeeded Professor G. J. BRUNNER as head of the department of mathematics.

Professor C. W. STROM, of Luther College, Decorah, Iowa, has been granted leave of absence for the current year and is studying at the University of Iowa.

At Fairmount College, Wichita, Kansas, Professor A. J. HOARE has resumed his work as head of the department, having recovered from a serious impairment of vision which he suffered four years ago. Professor H. G. TITT has been appointed professor of mathematics at Huron College, South Dakota. Mr. R. W. ELLIOTT has been appointed business manager and instructor of mathematics.

Mr. C. S. WHITNEY has been appointed professor of mathematics and head of the department at the State School of Mines, Miami, Oklahoma.

At the Panhandle Agricultural and Mechanical College, Goodwell, Oklahoma, a new mathematics department has been appointed, consisting of Mr. F. C. LEMON, of the University of Michigan, Mr. T. J. PRUET, of the N. W. State Teachers College, and Mr. HENRY HOUGHTON, of the University of Oklahoma.

Miss LIDA B. MAY, of the University of Texas, has been appointed to the chair of mathematics at the Kidd-Key College and Conservatory, Sherman, Texas.

Mr. R. Z. NEWSOM, head of the mathematics department at Washington College, Tenn., has succeeded Miss O. KATE CANNON as head of the department of mathematics at Rusk College, Rusk, Texas.

At Simmons College, Abilene, Texas, Professor A. E. CHANDLER has been made bursar, and Associate Professor J. E. BURNAM has been promoted to a full professorship.

Professor EMMA K. WHITON, of the University of Redlands, Redlands, Calif., has been appointed professor of mathematics at Mills College, Oakland, Calif.

NOTABLE PRIZE AWARD.

The committee on the award of the Cincinnati prize of the American Association for the Advancement of Science has adjudged the prize to Dr. L. E. DICKSON, professor of mathematics in the University of Chicago. The contributions for which the prize was awarded were presented before a joint session of Section A of the A. A. A. S., the American Mathematical Society, and the Mathematical Association of America on Friday afternoon, December 28, and before a session of the American Mathematical Society on Saturday afternoon, December 29. The paper presented at the Friday session was entitled "Algebras and their Arithmetics"; the papers presented at the Saturday session were entitled "On the Theory of Numbers and Generalized Quaternions," and "Quadratic Fields in which Factorization is Always Unique." Distinguished mathematicians from all parts of the country, present at these meetings, were unanimous in the opinion that the work presented by Professor Dickson constituted one of those outstanding contributions to the development of science which are made only at rare intervals. The committee on the award of the prize consisted of the following members: Dr. N. M. FENNEMAN, Professor of Geology, University of Cincinnati, Chairman; Dr. HENRY CREW, Professor of Physics, Northwestern University; Dr. C. H. PARKER, Professor of Zoölogy, Harvard University; Dr. E. W. WASHBURN, Professor of Chemistry, University of Illinois; Dr. G. T. MOORE, Director Missouri Botanical Gardens, St. Louis, Mo.

4

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